# Online Appendix for "Liquidity with High-Frequency Market Making"

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Let $\Delta_0(\mu)$ denote the bid-ask spread given $\mu$ before the arrival of new information, and $\Delta_s(\mu)$ denote the updated spread that will prevail on the market after the signal gets realized. We first show that imperfect substitution of the low-frequency market makers by high-frequency traders in liquidity provision can reduce liquidity during the time horizon when the speed advantage is irrelevant, i.e. when there are no news.

**Proposition 1.** *$\Delta_0(\mu)$ is concave in $\mu$ with $\Delta_0(0) = \Delta_0(1)$.*

*Proof.* The expected spread before the release of public information is given by

$$\Delta_0(\mu) = (1-\mu)\big(p^L_{ask} - p^L_{bid}\big) + \mu\big(p^H_{ask}(0) - p^H_{bid}(0)\big)$$

Indeed, if new information has not arrived, the bid-ask spread is either the one set by low-frequency traders or, if the high-frequency traders are on the market, it is the narrower spread quoted by high-frequency market makers. Since as of time 0 the probability of high-frequency traders being present is given by $\mu$, we have that expected uninformed spread as stated.

Now show that $\Delta_0(\mu)$ has the following properties:

(i) $\frac{\partial \Delta_0}{\partial \mu}\Big|_{\mu=0} > 0$ and $\frac{\partial \Delta_0}{\partial \mu}\Big|_{\mu=1} < 0$;

(ii) $\Delta_0$ is concave in $\mu$, i.e., $\frac{\partial^2 \Delta_0}{\partial \mu^2} < 0$;

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(iii) $\Delta_0(0) = \Delta_0(1)$.

Using the quotes of low-frequency traders we obtain

$$\begin{aligned}\Delta_0(\mu) = {} & \mu\big[E[\tilde{V}|\text{buy}] - E[\tilde{V}|\text{sell}]\big] \\ & + (1-\mu)\big[\omega_{ask}E[\tilde{V}|\text{buy}] - \omega_{bid}E[\tilde{V}|\text{sell}]\big] \\ & + (1-\mu)\big[(1-\omega_{ask})E[\tilde{V}|S^G, \text{buy}] - (1-\omega_{bid})E[\tilde{V}|S^B, \text{sell}]\big].\end{aligned}$$

Taking the first derivative of $\Delta_0$ with respect to $\mu$ and rearranging gives

$$\begin{aligned}\frac{\partial \Delta_0(\mu)}{\partial \mu} = {} & \big(1 - \omega_{ask} + (1-\mu)\frac{\partial \omega_{ask}}{\partial \mu}\big)\big[E[\tilde{V}|\text{buy}] - E[\tilde{V}|S^G, \text{buy}]\big] \\ & + \big(1 - \omega_{bid} + (1-\mu)\frac{\partial \omega_{bid}}{\partial \mu}\big)\big[-E[\tilde{V}|\text{sell}] + E[\tilde{V}|S^B, \text{sell}]\big].\end{aligned}$$

We then have $\frac{\partial \Delta_0(\mu)}{\partial \mu}\Big|_{\mu=1} = \big[E[\tilde{V}|\text{buy}] - E[\tilde{V}|S^G, \text{buy}]\big] + \big[-E[\tilde{V}|\text{sell}] + E[\tilde{V}|S^B, \text{sell}]\big] < 0$, because both parts of this expression are negative, and $\frac{\partial \Delta_0(\mu)}{\partial \mu}\Big|_{\mu=0} = \frac{\partial \omega_{ask}}{\partial \mu}\Big|_{\mu=0}\big[E[\tilde{V}|\text{buy}] - E[\tilde{V}|S^G, \text{buy}]\big] + \frac{\partial \omega_{bid}}{\partial \mu}\Big|_{\mu=0}\big[-E[\tilde{V}|\text{sell}] + E[\tilde{V}|S^B, \text{sell}]\big] > 0$ since $\frac{\partial \omega_{ask}}{\partial \mu} < 0$ and $\frac{\partial \omega_{bid}}{\partial \mu} < 0$. The second derivative of $\Delta_0$ with respect to $\mu$ is

$$\begin{aligned}\frac{\partial^2 \Delta_0(\mu)}{\partial \mu^2} = {} & \big(-2\frac{\partial \omega_{ask}}{\partial \mu} + (1-\mu)\frac{\partial^2 \omega_{ask}}{\partial \mu^2}\big)\big[E[\tilde{V}|\text{buy}] - E[\tilde{V}|S^G, \text{buy}]\big] \\ & + \big(-2\frac{\partial \omega_{bid}}{\partial \mu} + (1-\mu)\frac{\partial^2 \omega_{bid}}{\partial \mu^2}\big)\big[-E[\tilde{V}|\text{sell}] + E[\tilde{V}|S^B, \text{sell}]\big].\end{aligned}$$

It's straightforward to show that $\big(-2\frac{\partial \omega_{ask}}{\partial \mu} + (1-\mu)\frac{\partial^2 \omega_{ask}}{\partial \mu^2}\big) > 0$ and the same is true for the analogous expression in terms of $\omega_{bid}$. Since the expressions in the square brackets are negative we have that $\frac{\partial^2 \Delta_0(\mu)}{\partial \mu^2} < 0$ and this completes the proof. □

The narrowest pre-announcement spreads arise when the there are only low-frequency market makers ($\mu = 0$) or the high-frequency market makers are present on the market with certainty ($\mu = 1$). Worse scenarios arise when there is a nonzero probability that the high-frequency traders submitting quotes. This introduces potential losses for low-frequency traders. Faster traders can get informed and thus reduce their costs of getting "run over", which introduces additional losses that the low-frequency traders might bear (winner's curse). This situation, when the high-frequency traders don't fully replace the low-frequency market makers in providing liquidity, results in wider spreads.

If the public signal arrives before the market order, the bid-ask spread is expected to be updated since high-frequency trades react to the public news. As the next proposition states, post-announcement liquidity always improves with high-frequency trading. That is even with partial presence of high-frequency traders expected post-announcement liquidity is higher. This is a result of public information being incorporated into prices.

**Proposition 2.** $\Delta_s(\mu) \leq \Delta_s(0)$ *for all* $\mu > 0$*.*

*Proof.* The expected spread after the arrival of new information is given by

$$\Delta_s(\mu) = (1-\mu)\big(p^L_{ask} - p^L_{bid}\big) + \mu\big[P(S^G)\big(p^L_{ask} - p^H_{bid}(s)\big) + P(S^B)\big(p^H_{ask}(s) - p^L_{bid}\big)\big].$$

We show now that $\Delta_s(\mu) < \Delta_s(0), \quad \forall \mu > 0$. Indeed, considering the difference and rearranging we have

$$\begin{aligned}
&\Delta_s(\mu) - \Delta_s(0) \\
&= -\mu\big(p^L_{ask} - p^L_{bid}\big) + \mu\big[P(S^G)\big(p^L_{ask} - p^H_{bid}(s)\big) + P(S^B)\big(p^H_{ask}(s) - p^L_{bid}\big)\big] \\
&= \mu\big[P(S^G)\big(p^L_{ask} - p^H_{bid}(s)\big) + P(S^B)\big(p^H_{ask}(s) - p^L_{bid}\big) - (P(S^G) + P(S^B))\big(p^L_{ask} - p^L_{bid}\big)\big] \\
&= \mu\big[P(S^G)\big(p^L_{bid} - p^H_{bid}(s)\big) + P(S^B)\big(p^H_{ask}(s) - p^L_{ask}\big)\big].
\end{aligned}$$

This expression is negative for all $\mu$ since

$$\begin{aligned}
&p^L_{bid} - p^H_{bid}(s) \\
&= \omega_{bid} E[\tilde{V}|\text{sell}] + (1-\omega_{bid}) E[\tilde{V}|S^B, \text{sell}] - E[\tilde{V}|S^G, \text{sell}] \\
&= \omega_{bid} \underbrace{\big(E[\tilde{V}|\text{sell}] - E[\tilde{V}|S^G, \text{sell}]\big)}_{<0} + (1-\omega_{bid}) \underbrace{\big(E[\tilde{V}|S^B, \text{sell}] - E[\tilde{V}|S^G, \text{sell}]\big)}_{<0} < 0.
\end{aligned}$$

Similarly, we can show that $p^H_{ask}(s) - p^L_{ask} < 0$. Hence, the difference $\Delta_s(\mu) - \Delta_s(0)$ is negative for all $\mu > 0$. □

The bid-ask spread expected on the market at any point in time $t \in [0, 1]$ before submission of the market order can be written in terms of $\Delta_0$ as $\Delta_s$ as

$$\Delta_t = \Delta_0 P(t < s) + \Delta_s P(t \geq s).$$

We then have the following result about the average liquidity $\bar{\Delta}(\mu) \equiv \int_0^1 \Delta_t dt$.

**Proposition 3.** *For $\lambda < \bar{\lambda}$ we have $\bar{\Delta}(\mu) > \bar{\Delta}(0)$, that is average liquidity may deteriorate with the participation of high frequency traders for a range of $\lambda$.*

*Proof.* Integrating $\Delta_t$ over $[0,1]$ we obtain the following expression

$$\bar{\Delta}(\mu) = e^{-\lambda}\Delta_0 + (1-e^{-\lambda})\left[\Delta_0\left(\frac{1}{\lambda} - \frac{e^{-\lambda}}{(1-e^{-\lambda})}\right) + \Delta_s\left(1 - \frac{1}{\lambda} + \frac{e^{-\lambda}}{(1-e^{-\lambda})}\right)\right],$$

which can be further simplified to

$$\bar{\Delta}(\mu) = \Delta_0 \frac{1-e^{-\lambda}}{\lambda} + \Delta_s\left(1 - \frac{1-e^{-\lambda}}{\lambda}\right).$$

To show that liquidity can be worsened if high-frequency traders do not fully replace low-frequency traders in liquidity provision it's enough to show that we have a region of parameters for which $\bar{\Delta}(\mu) \geq \bar{\Delta}(0)$. Let's look at the difference

$$\bar{\Delta}(\mu) - \bar{\Delta}(0) = (\Delta_0(\mu) - \Delta(0))\frac{1-e^{-\lambda}}{\lambda} + (\Delta_s(\mu) - \Delta_s(0))\left(1 - \frac{1-e^{-\lambda}}{\lambda}\right).$$

We know from Propositions 1 and 2 in this Appendix that $\Delta_0(\mu) - \Delta(0) \geq 0$ whereas $\Delta_s(\mu) - \Delta_s(0) < 0$. When $\lambda \to 0$ we have $\frac{1-e^{-\lambda}}{\lambda} \to 1$ and thus the negative term goes to zero as well. For sufficiently low values of arrival rate of public announcement $\lambda$ we have $\bar{\Delta}(\mu) - \bar{\Delta}(0) > 0$.

We can also see that average liquidity is always better in the sure presence of high-frequency traders since for $\mu = 1$ we have $\bar{\Delta}(1) - \bar{\Delta}(0) \propto \Delta_s(1) - \Delta_s(0) < 0$. □

Perhaps more intuitively average spread can be rewritten as

$$\bar{\Delta} = \Delta_0 - P(s \leq 1)E\left[1 - s \mid s \leq 1\right](\Delta_0 - \Delta_s).$$

We thus see that is narrowed with (i) higher probability of payoff-relevant public information (high $P(s \leq 1)$; (ii) earlier expected arrival of such information (high $E\left[1 - s \mid s \leq 1\right]$; (iii) higher difference between pre-information and post-information spreads (high $\Delta_0 - \Delta_s$).