

# A Theory of Markovian Time Inconsistent Stochastic Control in Continuous Time \*

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## Abstract

In this paper, which is a continuation of the discrete time paper [4], we develop a theory for continuous time stochastic control problems which, in various ways, are time inconsistent in the sense that they do not admit a Bellman optimality principle. We study these problems within a game theoretic framework, and we look for Nash subgame perfect equilibrium points. Within the framework of a controlled SDE and a fairly general objective functional we derive an extension of the standard Hamilton-Jacobi-Bellman equation, in the form of a system of non-linear equations, for the determination for the equilibrium strategy as well as the equilibrium value function. As applications of the general theory we study non exponential discounting as well as a time inconsistent linear quadratic regulator. We also present a study of time inconsistency within the framework of a general equilibrium production economy of Cox-Ingersoll-Ross type [5].

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## 1 Introduction

The purpose of this paper is to study a class of stochastic control problems in continuous time, which have the property of being time-inconsistent in the sense that they do not allow for a Bellman optimality principle. As a consequence of this, the very concept of optimality becomes problematic, since a strategy which is optimal given a specific starting point in time and space, may be non-optimal when viewed from a later date and a different state. In this paper we attack, within the framework of a controlled SDE, a fairly general class of time inconsistent problems by using a game-theoretic approach, so instead of searching for *optimal* strategies we search for subgame perfect Nash *equilibrium* strategies. The paper presents a continuous time version of the discrete time theory developed in our previous paper [4]. Since we will build heavily on the discrete time paper, the reader is referred to that paper for motivating examples and for more detailed discussions on conceptual issues.

### 1.1 Previous literature

For a detailed discussion of the the game theoretic approach to time inconsistency using Nash equilibrium points as above the reader is referred to [4]. A list of some of the most important papers on the subject is given by [2], [6], [7], [8], [9], [13],[16], [17], [18], [19], and [20].

All the papers above deal with particular model choices, and different authors use different methods in order to solve the problems. To our knowledge, the present paper, which is the continuous time part of the working paper [3], is the first attempt to derive a reasonably *general* (albeit Markovian) theory of time inconsistent control in continuous time. We would, however, like to stress that for the present paper we have been greatly inspired by [2], [7], and [8].

## 1.2 Structure of the paper

The structure of the paper is roughly as follows.

- In Section 2 we present the basic setup, and in Section 3 we discuss the concept of *equilibrium*. This concept replaces, in our setting, the optimality concept for a standard stochastic control problem, and in Definition 3.2 we give a precise definition of the equilibrium control and the equilibrium value function.
- Since the equilibrium concept in continuous time is quite delicate, we build the continuous time theory on the discrete time theory previously developed in [4]. In Section 4 we start to study the continuous time problem by going to the limit for a discretized problem, and using the results from [4]. This leads to an extension of the standard HJB equation to a system of equations with an embedded static optimization problem. The limiting procedure described above is done in an informal manner. It is largely heuristic, and it thus remains to clarify precisely how the derived extended HJB system is related to the precisely defined equilibrium problem under consideration.
- The needed clarification is in fact delivered in Section 5. In Theorem 5.1, which is the main theoretical result of the paper, we give a precise statement and proof of a verification theorem. This theorem says that a solution to the extended HJB system does indeed deliver the equilibrium control and equilibrium value function to our original problem.
- In Section 6 the results of Section 5 are extended to a more general reward functional.
- Section 7 is devoted to important special cases and various extensions.
- In Section 8 we prove that for every time inconsistent problem there exists an associated standard (i.e. time consistent) control problem which in very strong sense is equivalent to the time inconsistent problem.
- In Sections 9-10 we study some examples to illustrate how the theory works in a concrete case.
- Section 11 is devoted to a rather detailed study of a general equilibrium model for a production economy with time inconsistent preferences.

## 2 The model

We now turn to the formal continuous time theory. In order to present this we need some input data.

**Definition 2.1** *The following objects are given exogenously.*

1. A drift mapping  $\mu : R_+ \times R^n \times R^k \rightarrow R^n$ .
2. A diffusion mapping  $\sigma : R_+ \times R^n \times R^k \rightarrow M(n, d)$ , where  $M(n, d)$  denotes the set of all  $n \times d$  matrices.
3. A control constraint mapping  $U : R_+ \times R^n \rightarrow 2^{R^k}$
4. A mapping  $F : R^n \times R^n \rightarrow R$ .
5. A mapping  $G : R^n \times R^n \rightarrow R$ .

We now consider, on the time interval  $[0, T]$ , a controlled SDE of the form

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t. \quad (1)$$

where the state process  $X$  is  $n$ -dimensional, the Wiener process  $W$  is  $d$ -dimensional, and the control process  $u$  is  $k$ -dimensional, with the constraint  $u_t \in U(t, X_t)$ .

Loosely speaking our object is to maximize, for every initial point  $(t, x)$ , a reward functional of the form

$$E_{t,x} [F(x, X_T)] + G(x, E_{t,x} [X_T]). \quad (2)$$

This functional is not of a form which is suitable for dynamic programming, and it will be discussed in detail below, but first we need to specify our class of controls. In this paper we restrict the controls to **admissible feedback control laws**.

**Definition 2.2** An **admissible control law** is a map  $\mathbf{u} : [0, T] \times R^n \rightarrow R^k$  satisfying the following conditions:

1. For each  $(t, x) \in [0, T] \times R^n$  we have  $\mathbf{u}(t, x) \in U(t, x)$ .
2. For each initial point  $(s, y) \in [0, T] \times R^n$  the SDE

$$dX_t = \mu(t, X_t, \mathbf{u}(t, X_t))dt + \sigma(t, X_t, \mathbf{u}(t, X_t))dW_t. \quad (3)$$

has a unique strong solution denoted by  $X^{\mathbf{u}}$ .

The class of admissible control laws is denoted by  $\mathbf{U}$ . We will sometimes use the notation  $\mathbf{u}_t(x)$  instead of  $\mathbf{u}(t, x)$ .

We now go on to define the controlled infinitesimal generator of the SDE above. In the present paper we use the (somewhat non-standard) convention that the infinitesimal operator acts on the time variable as well as on the space variable, so it includes the term  $\frac{\partial}{\partial t}$ . The reason for this notational convention is to have formal similarity of the continuous time theory with the discrete time theory of [4]. It will facilitate some arguments below considerably.

**Definition 2.3** Consider the SDE (1), and let  $'$  denote matrix transpose.

- For any fixed  $u \in R^k$ , the functions  $\mu^u$ ,  $\sigma^u$  and  $C^u$  are defined by

$$\begin{aligned}\mu^u(t, x) &= \mu(t, x, u), \\ \sigma^u(t, x) &= \sigma(t, x, u), \\ C^u(t, x) &= \sigma(t, x, u)\sigma(t, x, u)'.\end{aligned}$$

- For any admissible control law  $\mathbf{u}$ , the functions  $\mu^{\mathbf{u}}$ ,  $\sigma^{\mathbf{u}}$ ,  $C^{\mathbf{u}}(t, x)$  are defined by

$$\begin{aligned}\mu^{\mathbf{u}}(t, x) &= \mu(t, x, \mathbf{u}(t, x)), \\ \sigma^{\mathbf{u}}(t, x) &= \sigma(t, x, \mathbf{u}(t, x)), \\ C^{\mathbf{u}}(t, x) &= \sigma(t, x, \mathbf{u}(t, x))\sigma(t, x, \mathbf{u}(t, x))'.\end{aligned}$$

- For any fixed  $u \in R^k$ , the operator  $\mathbf{A}^u$  is defined by

$$\mathbf{A}^u = \frac{\partial}{\partial t} + \sum_{i=1}^n \mu_i^u(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_{ij}^u(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

- For any admissible control law  $\mathbf{u}$ , the operator  $\mathbf{A}^{\mathbf{u}}$  is defined by

$$\mathbf{A}^{\mathbf{u}} = \frac{\partial}{\partial t} + \sum_{i=1}^n \mu_i^{\mathbf{u}}(t, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n C_{ij}^{\mathbf{u}}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

### 3 Problem formulation

In order to formulate our problem we need an objective functional. We thus consider the two functions  $F$  and  $G$  from Definition 2.1.

**Definition 3.1** For a fixed  $(t, x) \in [0, T] \times R^n$ , and a fixed admissible control law  $\mathbf{u}$ , the corresponding **reward functional**  $J$  is defined by

$$J(t, x, \mathbf{u}) = E_{t,x} [F(x, X_T^{\mathbf{u}})] + G(x, E_{t,x} [X_T^{\mathbf{u}}]). \quad (4)$$

**Remark 3.1** In Section 6 we will consider a more general reward functional. The restriction to the functional (4) above is done in order to minimize the notational complexity of the derivations below, which otherwise would be somewhat messy.

In order to have a non degenerate problem we need a formal integrability assumption.

**Assumption 3.1** We assume that for each initial point  $(t, x) \in [0, T] \times R^n$ , and each admissible control law  $\mathbf{u}$ , we have

$$E_{t,x} [|F(x, X_T^{\mathbf{u}})|] < \infty, \quad E_{t,x} [|X_T^{\mathbf{u}}|] < \infty$$

and hence

$$G(x, E_{t,x} [X_T^{\mathbf{u}}]) < \infty.$$

Our objective is loosely that of maximizing  $J(t, x, \mathbf{u})$  for each  $(t, x)$ , but conceptually this turns out to be far from trivial, so instead of *optimal* controls we will study *equilibrium* controls. The equilibrium concept is made precise in Definition 3.2 below, but in order to motivate that definition we need a brief discussion concerning the reward functional above.

We immediately note that, compared to a standard optimal control problem, the family of reward functionals above are not connected by a Bellman optimality principle. The reasons for this are as follows:

- The present state  $x$  appears in the function  $F$ .
- In the second term we have (even apart from the appearance of the present state  $x$ ), a nonlinear function  $G$  operating on the expected value  $E_{t,x} [X_T^{\mathbf{u}}]$ .

Since we do not have a Bellman optimality principle it is in fact unclear what we would mean by the term “optimal”, since the optimality concept would differ at different initial times  $t$  and for different initial states  $x$ .

The approach taken in this paper is to look at the problem from a game theoretic perspective, and look for subgame perfect Nash equilibrium points. This will be given a precise definition below, but loosely speaking we view the game as follows:

- Consider a non-cooperative game, where we have one player for each point in time  $t$ . We refer to this player as “Player  $t$ ”.
- For each fixed  $t$ , Player  $t$  can only control the process  $X$  exactly at time  $t$ . He/she does that by choosing a control function  $\mathbf{u}(t, \cdot)$ , so the action taken at time  $t$  with state  $X_t$  is given by  $\mathbf{u}(t, X_t)$ .
- Gluing together the control functions for all players we thus have a feedback control law  $\mathbf{u} : [0, T] \times R^n \rightarrow R^k$ .
- Given the feedback law  $\mathbf{u}$ , the reward to Player  $t$  is given by the reward functional

$$J(t, x, \mathbf{u}) = E_{t,x} [F(x, X_T^{\mathbf{u}})] + G(x, E_{t,x} [X_T^{\mathbf{u}}])$$

An informal (and slightly naive) definition of an equilibrium for this game would be to say that a feedback control law  $\hat{\mathbf{u}}$  is a subgame perfect Nash equilibrium if, for each  $t$ , it has the following property:

- If for each each  $s > t$ , Player  $s$  chooses the the control  $\hat{\mathbf{u}}(s, \cdot)$ , then it is optimal for Player  $t$  to choose  $\hat{\mathbf{u}}(t, \cdot)$ .

A definition like this works well in discrete time, but in continuous time this is not a bona fide definition. Since Player  $t$  can only choose the control  $u_t$  exactly at time  $t$ , he only influences the control on a time set of Lebesgue measure zero, so for a controlled SDE of the form (1) the control chosen by an individual player will have no effect whatsoever on the dynamics of the process. We thus

need another definition of the equilibrium concept, and we will in fact follow an approach first taken by [7] and [8]. The formal definition of equilibrium is now as follows.

**Definition 3.2** Consider an admissible control law  $\hat{\mathbf{u}}$  (informally viewed as a candidate equilibrium law). Choose an arbitrary admissible control law  $\mathbf{u} \in \mathbf{U}$  and a fixed real number  $h > 0$ . Also fix an arbitrarily chosen initial point  $(t, x)$ . Define the control law  $\mathbf{u}_h$  by

$$\mathbf{u}_h(s, y) = \begin{cases} \mathbf{u}(s, y), & \text{for } t \leq s < t + h, \quad y \in R^n, \\ \hat{\mathbf{u}}(s, y), & \text{for } t + h \leq s \leq T, \quad y \in R^n. \end{cases}$$

If

$$\liminf_{h \rightarrow 0} \frac{J(t, x, \hat{\mathbf{u}}) - J(t, x, \mathbf{u}_h)}{h} \geq 0,$$

for all  $\mathbf{u} \in \mathbf{U}$ , we say that  $\hat{\mathbf{u}}$  is an **equilibrium** control law. Corresponding to the equilibrium law  $\hat{\mathbf{u}}$  we define the **equilibrium value function**  $V$  by

$$V(t, x) = J(t, x, \hat{\mathbf{u}}).$$

We will sometimes refer to this as an *intrapersonal equilibrium*, since it can be viewed as a game between different future manifestations of you own preferences.

**Remark 3.2** This is our continuous time formalization of the corresponding discrete time equilibrium concept.

Note the necessity of dividing by  $h$ , since for most models we trivially would have

$$\lim_{h \rightarrow 0} \{J(t, x, \hat{\mathbf{u}}) - J(t, x, \mathbf{u}_h)\} = 0.$$

We also note that we do not get a perfect correspondence with the discrete time equilibrium concept, since if the limit above equals zero for all  $\mathbf{u} \in \mathbf{U}$ , it is not clear that this corresponds to a maximum or just to a stationary point.

**Remark 3.3** There may exist multiple equilibria, so the equilibrium value function should strictly be denoted by  $V(t, x, \hat{\mathbf{u}})$  but we use  $V(t, x)$  for ease of notation.

## 4 An informal derivation of the extended HJB equation

We now assume that there exists an equilibrium control law  $\hat{\mathbf{u}}$  (not necessarily unique) and we go on to derive an extension of the standard Hamilton-Jacobi-Bellman (henceforth HJB) equation for the determination of the corresponding value function  $V$ . To clarify the logical structure of the derivation we outline our strategy as follows.



- We discretize (to some extent) the continuous time problem. We then use our results from discrete time theory to obtain a discretized recursion for  $\hat{\mathbf{u}}$  and we then let the time step tend to zero.
- In the limit we obtain our continuous time extension of the HJB equation. Not surprisingly it will in fact be a system of equations.
- In the discretizing and limiting procedure we mainly rely on informal heuristic reasoning. In particular we have do **not** claim that the derivation is a rigorous one. The derivation is, from a logical point of view, only of motivational value.
- In Section 5 we then go on to show that our (informally derived) extended HJB equation is in fact the “correct” one, by proving a rigorous verification theorem.

#### 4.1 Deriving the equation

In this section we will, in an informal and heuristic way, derive a continuous time extension of the HJB equation. Note again that we have no claims to rigor in the derivation, which is only motivational. To this end we assume that there exists an equilibrium law  $\hat{\mathbf{u}}$  and we argue as follows.

- Choose an arbitrary initial point  $(t, x)$ . Also choose a “small” time increment  $h > 0$  and an arbitrary admissible control  $\mathbf{u}$ .
- Define the control law  $\mathbf{u}_h$  on the time interval  $[t, T]$  by

$$\mathbf{u}_h(s, y) = \begin{cases} \mathbf{u}(s, y), & \text{for } t \leq s < t + h, \quad y \in R^n, \\ \hat{\mathbf{u}}(s, y), & \text{for } t + h \leq s \leq T, \quad y \in R^n. \end{cases}$$

- If now  $h$  is “small enough” we expect to have

$$J(t, x, \mathbf{u}_h) \leq J(t, x, \hat{\mathbf{u}}),$$

and in the limit as  $h \rightarrow 0$  we should have equality if  $\mathbf{u}(t, x) = \hat{\mathbf{u}}(t, x)$ .

We now refer to the discrete time results, as well as the notation, from Theorem 3.13 of [4], with  $n$  and  $n + 1$  replaced by  $t$  and  $t + h$ . We then obtain the inequality

$$(\mathbf{A}_h^{\mathbf{u}}V)(t, x) - (\mathbf{A}_h^{\mathbf{u}}f)(t, x, x) + (\mathbf{A}_h^{\mathbf{u}}f^x)(t, x) - \mathbf{A}_h^{\mathbf{u}}(G \diamond g)(t, x) + (\mathbf{H}_h^{\mathbf{u}}g)(t, x) \leq 0$$

Here we have used the following notation from [4].

- For any fixed  $y \in R^n$  the mapping  $f^y : [0, T] \times R^n \rightarrow R$  is defined by

$$f^y(t, x) = E_{t,x} [F(y, X_T^{\hat{\mathbf{u}}})]$$

- The function  $f : [0, T] \times R^n \times R^n \rightarrow R$  is defined by

$$f(t, x, y) = f^y(t, x).$$

We will also, with a slight abuse of notation, denote the entire family of functions  $\{f^y; y \in R^n\}$  by  $f$ .

- For any function  $k(t, x)$  the operator  $\mathbf{A}_h^{\mathbf{u}}$  is defined by

$$(\mathbf{A}_h^{\mathbf{u}}k)(t, x) = E_{t,x} [k(t+h, X_{t+h}^{\mathbf{u}})] - k(t, x).$$

- The function  $g : [0, T] \times R^n \rightarrow R^n$  is defined by

$$g(t, x) = E_{t,x} [X_T^{\mathbf{u}}].$$

- The function  $G \diamond g$  is defined by

$$(G \diamond g)(t, x) = G(x, g(t, x))$$

- The term  $\mathbf{H}_h^{\mathbf{u}}g$  is defined by

$$(\mathbf{H}_h^{\mathbf{u}}g)(t, x) = G(x, E_{t,x} [g(t+h, X_{t+h}^{\mathbf{u}})]) - G(x, g(t, x)).$$

We now divide the inequality by  $h$  and let  $h$  tend to zero. The the operator  $\mathbf{A}_h^{\mathbf{u}}$  will converge to the infinitesimal operator  $\mathbf{A}^u$ , where  $u = \mathbf{u}(t, x)$  but the limit of  $h^{-1}(\mathbf{H}_h^{\mathbf{u}}g)(t, x)$  requires closer investigation.

From the definition of the infinitesimal operator we have the approximation

$$E_{t,x} [g(t+h, X_{t+h}^{\mathbf{u}})] = g(t, x) + \mathbf{A}^u g(t, x) + o(h),$$

and using a standard Taylor approximation for  $G^x$  we obtain

$$G(x, E_{t,x} [g(t+h, X_{t+h}^{\mathbf{u}})]) = G(x, g(t, x)) + G_y(x, g(t, x)) \cdot \mathbf{A}^u g(t, x) + o(h),$$

where

$$G_y(x, y) = \frac{\partial G}{\partial y}(x, y).$$

We thus obtain

$$\lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{H}_h^{\mathbf{u}}g)(t, x) = G_y(x, g(t, x)) \cdot \mathbf{A}^u g(t, x).$$

Collecting all results we arrive at our proposed extension of the HJB equation. To stress the fact that the arguments above are largely informal we state the equation as a definition rather than as proposition.

**Definition 4.1** *The extended HJB system of equations for  $V$ ,  $f$ , and  $g$ , is defined as follows.*

1. The function  $V$  is determined by

$$\begin{aligned} \sup_{u \in U(t,x)} \{ & (\mathbf{A}^u V)(t,x) - (\mathbf{A}^u f)(t,x,x) + (\mathbf{A}^u f^x)(t,x) \\ & - \mathbf{A}^u (G \diamond g)(t,x) + (\mathbf{H}^u g)(t,x) \} = 0, \quad 0 \leq t \leq T, \\ V(T,x) & = F(x,x) + G(x,x). \end{aligned} \quad (5)$$

2. For every fixed  $y \in R^n$  the function  $(t,x) \mapsto f^y(t,x)$  is defined by

$$\begin{aligned} \mathbf{A}^{\hat{u}} f^y(t,x) & = 0, \quad 0 \leq t \leq T, \\ f^y(T,x) & = F(y,x). \end{aligned} \quad (6)$$

3. The function  $g$  is defined by

$$\begin{aligned} \mathbf{A}^{\hat{u}} g(t,x) & = 0, \quad 0 \leq t \leq T, \\ g(T,x) & = x. \end{aligned} \quad (7)$$

We now have some comments on the extended HJB system.

- The first point to notice is that we have a **system** of equations (5)-(7) for the simultaneous determination of  $V$ ,  $f$  and  $g$ .
- In the expressions above,  $\hat{u}$  always denotes the control law which realizes the supremum in the first equation.
- The equations (6)-(7) are the Kolmogorov backward equations for the expectations

$$\begin{aligned} f^y(t,x) & = E_{t,x} [F(y, X_T^{\hat{u}})], \\ g(t,x) & = E_{t,x} [X_T^{\hat{u}}]. \end{aligned}$$

- In order to solve the  $V$ -equation we need to know  $f$  and  $g$  but these are determined by the equilibrium control law  $\hat{u}$ , which in turn is determined by the *sup*-part of the  $V$ -equation.
- We have used the notation

$$\begin{aligned} f(t,x,y) & = f^y(t,x) \\ (G \diamond g)(t,x) & = G(x, g(t,x)), \\ \mathbf{H}^u g(t,x) & = G_y(x, g(t,x)) \cdot \mathbf{A}^u g(t,x), \\ G_y(x,y) & = \frac{\partial G}{\partial y}(x,y). \end{aligned}$$

- The operator  $\mathbf{A}^u$  only operates on variables within parenthesis. Thus the expression  $(\mathbf{A}^u f)(t,x,x)$  is interpreted as  $(\mathbf{A}^u h)(t,x)$  with  $h$  defined by  $h(t,x) = f(t,x,x)$ . In the expression  $(\mathbf{A}^u f^y)(t,x)$  the operator does not act on the upper case index  $y$ , which is viewed as a fixed parameter. Similarly, in the expression  $(\mathbf{A}^u f^x)(t,x)$ , the operator only acts on the variables  $t,x$  within the parenthesis, and does not act on the upper case index  $x$ .

- In the case when  $F(x, y)$  does not depend upon  $x$ , and there is no  $G$  term, the problem trivializes to a standard time consistent problem. The terms  $(\mathbf{A}^u f)(t, x, x) + (\mathbf{A}^u f^x)(t, x)$  in the  $V$ -equation cancel, and the system reduces to the standard Bellman equation

$$\begin{aligned}(\mathbf{A}^u V)(t, x) &= 0, \\ V(T, x) &= F(x).\end{aligned}$$

- We note that the  $g$  function above appears, in a more restricted framework, already in [2], [7], and [8].

## 4.2 Existence and uniqueness

The task of proving existence and/or uniqueness of solutions to the extended HJB system seems (at least to us) to be technically extremely difficult. We have no idea about how to proceed so we leave it for future research. It is thus very much an open problem.

## 5 A Verification Theorem

As we have noted above, the derivation of the continuous time extension of the HJB equation in the previous section was very informal. Nevertheless, it seems reasonable to expect that the system in Definition 4.1 will indeed determine the equilibrium value function  $V$ , but so far nothing has been formally proved. The following two conjectures are, however, natural.

1. Assume that there exists an equilibrium law  $\hat{\mathbf{u}}$  and that  $V$  is the corresponding value function. Assume furthermore that  $V$  is in  $C^{1,2}$ . Define  $f^y$  and  $g$  by

$$f^y(t, x) = E_{t,x} [F(y, X_T^{\hat{\mathbf{u}}})], \quad (8)$$

$$g(t, x) = E_{t,x} [X_T^{\hat{\mathbf{u}}}] . \quad (9)$$

We then conjecture that  $V$  satisfies the extended HJB system and that  $\hat{\mathbf{u}}$  realizes the supremum in the equation.

2. Assume that  $V$ ,  $f$ , and  $g$  solves the extended HJB system and that the supremum in the  $V$ -equation is attained for every  $(t, x)$ . We then conjecture that there exists an equilibrium law  $\hat{\mathbf{u}}$ , and that it is given by the maximizing  $u$  in the  $V$ -equation. Furthermore we conjecture that  $V$  is the corresponding equilibrium value function, and  $f$  and  $g$  allow for the interpretations (8)-(9).

In this paper we do not attempt to prove the first conjecture. Even for a standard time consistent control problem within an SDE framework, it is well known that this is technically quite complicated, and it typically requires the

theory of viscosity solutions. We will, however, prove the second conjecture. This obviously has the form of a verification result, and from standard theory we would expect that it can be proved with a minimum of technical complexity. We now give the precise formulation and proof of the verification theorem, but first we need to define a function space.

**Definition 5.1** Consider an arbitrary admissible control  $\mathbf{u} \in \mathbf{U}$ . A function  $h : R_+ \times R^n \rightarrow R$  is said to belong to the space  $L^2(X^{\mathbf{u}})$  if it satisfies the condition

$$E_{t,x} \left[ \int_t^T \|h_x(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}})\|^2 ds \right] < \infty \quad (10)$$

for every  $(t, x)$ . In this expression  $h_x$  denotes the gradient of  $h$  in the  $x$ -variable.

We can now state and prove the main result of the present paper.

**Theorem 5.1 (Verification Theorem)** Assume that (for all  $y$ ) the functions  $V(t, x)$ ,  $f^y(t, x)$ ,  $g(t, x)$ , and  $\hat{\mathbf{u}}(t, x)$  have the following properties.

1.  $V$ ,  $f^y$ , and  $g$  solves the extended HJB system in Definition 4.1.
2.  $V(t, x)$ , and  $g(t, x)$  are smooth in the sense that they are in  $C^{1,2}$ , and  $f(t, x, y)$  is in  $C^{1,2,2}$ .
3. The function  $\hat{\mathbf{u}}$  realizes the supremum in the  $V$  equation, and  $\hat{\mathbf{u}}$  is an admissible control law.
4.  $V$ ,  $f^y$ ,  $g$ , and  $G \diamond g$ , as well as the function  $(t, x) \mapsto f(t, x, x)$  all belong to the space  $L^2(X^{\hat{\mathbf{u}}})$ .

Then  $\hat{\mathbf{u}}$  is an equilibrium law, and  $V$  is the corresponding equilibrium value function. Furthermore,  $f$  and  $g$  can be interpreted according to (8)-(9).

**Proof.** The proof consists of two steps:

- We start by showing that  $f$  and  $g$  have the interpretations (8)-(9) and that  $V$  is the value function corresponding to  $\hat{\mathbf{u}}$ , i.e. that  $V(t, x) = J(t, x, \hat{\mathbf{u}})$ .
- In the second step we then prove that  $\hat{\mathbf{u}}$  is indeed an equilibrium control law.

To show that  $f$  and  $g$  have the interpretations (8)-(9) we apply the Ito formula to the processes  $f^y(s, X_s^{\hat{\mathbf{u}}})$  and  $g(s, X_s^{\hat{\mathbf{u}}})$ . Using (6)-(7) and the assumed integrability conditions for  $f^y$  and  $g$ , it follows that the processes  $f^y(s, X_s^{\hat{\mathbf{u}}})$  and  $g(s, X_s^{\hat{\mathbf{u}}})$  are martingales, so from the boundary conditions for  $f^y$  and  $g$  we obtain our desired representations of  $f^y$  and  $g$  as

$$f^y(t, x) = E_{t,x} [F(y, X_T^{\hat{\mathbf{u}}})], \quad (11)$$

$$g(t, x) = E_{t,x} [X_T^{\hat{\mathbf{u}}}] . \quad (12)$$

To show that  $V(t, x) = J(t, x, \hat{\mathbf{u}})$ , we use the  $V$  equation (5) to obtain:

$$\begin{aligned} (\mathbf{A}^{\hat{\mathbf{u}}}V)(t, x) - (\mathbf{A}^{\hat{\mathbf{u}}}f)(t, x, x) + (\mathbf{A}^{\hat{\mathbf{u}}}f^x)(t, x) \\ - \mathbf{A}^{\hat{\mathbf{u}}}(G \diamond g)(t, x) + (\mathbf{H}^{\hat{\mathbf{u}}}g)(t, x) = 0, \end{aligned} \quad (13)$$

where

$$\mathbf{H}^{\hat{\mathbf{u}}}g(t, x) = G_y(x, g(t, x)) \cdot \mathbf{A}^{\hat{\mathbf{u}}}g(t, x).$$

Since  $f$ , and  $g$  satisfies (6)-(7), we have

$$\begin{aligned} (\mathbf{A}^{\hat{\mathbf{u}}}f^x)(t, x) &= 0, \\ \mathbf{A}^{\hat{\mathbf{u}}}g(t, x) &= 0, \end{aligned}$$

so (13) takes the form

$$(\mathbf{A}^{\hat{\mathbf{u}}}V)(t, x) = (\mathbf{A}^{\hat{\mathbf{u}}}f)(t, x, x) + \mathbf{A}^{\hat{\mathbf{u}}}(G \diamond g)(t, x) \quad (14)$$

for all  $t$  and  $x$ .

We now apply the Ito formula to the process  $V(s, X_s^{\hat{\mathbf{u}}})$ . Integrating and taking expectations gives us

$$E_{t,x} [V(T, X_T^{\hat{\mathbf{u}}})] = V(t, x) + E_{t,x} \left[ \int_t^T \mathbf{A}^{\hat{\mathbf{u}}}V(s, X_s^{\hat{\mathbf{u}}}) ds \right],$$

where the stochastic integral part has vanished because of the integrability condition  $V \in L^2(X^{\hat{\mathbf{u}}})$ . Using (14) we thus obtain

$$E_{t,x} [V(T, X_T^{\hat{\mathbf{u}}})] = V(t, x) + E_{t,x} \left[ \int_t^T \mathbf{A}^{\hat{\mathbf{u}}}f(s, X_s^{\hat{\mathbf{u}}}, X_s^{\hat{\mathbf{u}}}) ds \right] + E_{t,x} \left[ \int_t^T \mathbf{A}^{\hat{\mathbf{u}}}(G \diamond g)(s, X_s^{\hat{\mathbf{u}}}) ds \right].$$

In the same way we obtain

$$\begin{aligned} E_{t,x} \left[ \int_t^T \mathbf{A}^{\hat{\mathbf{u}}}f(s, X_s^{\hat{\mathbf{u}}}, X_s^{\hat{\mathbf{u}}}) ds \right] &= E_{t,x} [f(T, X_T^{\hat{\mathbf{u}}}, X_T^{\hat{\mathbf{u}}})] - f(t, x, x), \\ E_{t,x} \left[ \int_t^T \mathbf{A}^{\hat{\mathbf{u}}}(G \diamond g)(s, X_s^{\hat{\mathbf{u}}}) ds \right] &= E_{t,x} [G(X_T, g(T, X_T^{\hat{\mathbf{u}}}))] - G(x, g(t, x)). \end{aligned}$$

Using this and the boundary conditions for  $V$ ,  $f$ , and  $g$  we get

$$\begin{aligned} E_{t,x} [F(X_T^{\hat{\mathbf{u}}}, X_T^{\hat{\mathbf{u}}}) + G(X_T^{\hat{\mathbf{u}}}, X_T^{\hat{\mathbf{u}}})] &= V(t, x) + E_{t,x} [F(X_T^{\hat{\mathbf{u}}}, X_T^{\hat{\mathbf{u}}})] - f(t, x, x) \\ &+ E_{t,x} [G(X_T^{\hat{\mathbf{u}}}, X_T^{\hat{\mathbf{u}}})] - G(x, g(t, x)), \end{aligned}$$

i.e.

$$V(t, x) = f(t, x, x) + G(x, g(t, x)). \quad (15)$$

Plugging (11)-(12) into (15) we get

$$V(t, x) = E_{t,x} [F(x, X_T^{\hat{\mathbf{u}}})] + G(x, E_{t,x} [X_T^{\hat{\mathbf{u}}})].$$

so we obtain the desired result

$$V(t, x) = J(t, x, \hat{\mathbf{u}}).$$

We now go on to show that  $\hat{\mathbf{u}}$  is indeed an equilibrium law, but first we need a small temporary definition. For any admissible control law  $\mathbf{u}$  we define  $f^{\mathbf{u}}$  and  $g^{\mathbf{u}}$  by

$$\begin{aligned} f^{\mathbf{u}}(t, x, y) &= E_{t,x} [F(y, X_T^{\mathbf{u}})], \\ g^{\mathbf{u}}(t, x) &= E_{t,x} [X_T^{\mathbf{u}}]. \end{aligned}$$

so, in particular we have  $f = f^{\hat{\mathbf{u}}}$  and  $g = g^{\hat{\mathbf{u}}}$ . For any  $h > 0$ , and any admissible control law  $\mathbf{u} \in \mathbf{U}$ , we now construct the control law  $\mathbf{u}_h$  defined in Definition 3.2. From Lemma 3.3 and Lemma 8.8 in [4], applied to the points  $t$  and  $t + h$  we obtain

$$\begin{aligned} J(t, x, \mathbf{u}_h) &= E_{t,x} [J(t + h, X_{t+h}^{\mathbf{u}_h}, \mathbf{u}_h)] \\ &\quad - \{E_{t,x} [f^{\mathbf{u}_h}(t + h, X_{t+h}^{\mathbf{u}_h}, X_{t+h}^{\mathbf{u}_h})] - E_{t,x} [f^{\mathbf{u}_h}(t + h, X_{t+h}^{\mathbf{u}_h}, x)]\} \\ &\quad - \{E_{t,x} [G(X_{t+h}^{\mathbf{u}_h}, g^{\mathbf{u}_h}(t + h, X_{t+h}^{\mathbf{u}_h}))] - G(x, E_{t,x} [g^{\mathbf{u}_h}(t + h, X_{t+h}^{\mathbf{u}_h}))])\}. \end{aligned}$$

Since  $\mathbf{u}_h = \mathbf{u}$  on  $[t, t + h]$ , we have  $X_{t+h}^{\mathbf{u}_h} = X_{t+h}^{\mathbf{u}}$ , and since  $\mathbf{u}_h = \hat{\mathbf{u}}$  on  $[t + h, T]$  we have

$$\begin{aligned} J(t + h, X_{t+h}^{\mathbf{u}_h}, \mathbf{u}_h) &= V(t + h, X_{t+h}^{\mathbf{u}}), \\ f^{\mathbf{u}_h}(t + h, X_{t+h}^{\mathbf{u}_h}, X_{t+h}^{\mathbf{u}_h}) &= f(t + h, X_{t+h}^{\mathbf{u}}, X_{t+h}^{\mathbf{u}}), \\ f^{\mathbf{u}_h}(t + h, X_{t+h}^{\mathbf{u}_h}, x) &= f(t + h, X_{t+h}^{\mathbf{u}}, x), \\ g^{\mathbf{u}_h}(t + h, X_{t+h}^{\mathbf{u}_h}) &= g(t + h, X_{t+h}^{\mathbf{u}}), \end{aligned}$$

so we obtain

$$\begin{aligned} J(t, x, \mathbf{u}_h) &= E_{t,x} [V(t + h, X_{t+h}^{\mathbf{u}})] \\ &\quad - \{E_{t,x} [f(t + h, X_{t+h}^{\mathbf{u}}, X_{t+h}^{\mathbf{u}})] - E_{t,x} [f(t + h, X_{t+h}^{\mathbf{u}}, x)]\} \\ &\quad - \{E_{t,x} [G(X_{t+h}^{\mathbf{u}}, g(t + h, X_{t+h}^{\mathbf{u}}))] - G(x, E_{t,x} [g(t + h, X_{t+h}^{\mathbf{u}})])\}. \end{aligned}$$

Furthermore, from the  $V$ -equation (5) we have

$$\begin{aligned} (\mathbf{A}^u V)(t, x) - (\mathbf{A}^u f)(t, x, x) + (\mathbf{A}^u f^x)(t, x) \\ - \mathbf{A}^u (G \diamond g)(t, x) + (\mathbf{H}^u g)(t, x) \leq 0, \end{aligned}$$

where we have used the notation  $u = \mathbf{u}(t, x)$ . This gives us

$$\begin{aligned} E_{t,x} [V(t + h, X_{t+h}^{\mathbf{u}})] - V(t, x) - \{E_{t,x} [f(t, X_{t+h}^{\mathbf{u}}, X_{t+h}^{\mathbf{u}})] - f(t, x, x)\} \\ + E_{t,x} [f(t, X_{t+h}^{\mathbf{u}}, x)] - f(t, x, x) \\ - E_{t,x} [G(t + h, g(t + h, X_{t+h}^{\mathbf{u}}))] + G(x, g(t, x)) \\ + G(x, E_{t,x} [g(t + h, X_{t+h}^{\mathbf{u}})]) - G(x, g(t, x)) \leq o(h), \end{aligned}$$

or, after simplification,

$$\begin{aligned} V(t, x) &\geq E_{t,x} [V(t+h, X_{t+h}^{\mathbf{u}})] - E_{t,x} [f(t, X_{t+h}^{\mathbf{u}}, X_{t+h}^{\mathbf{u}})] + E_{t,x} [f(t, X_{t+h}^{\mathbf{u}}, x)] \\ &\quad - E_{t,x} [G(t+h, g(t+h, X_{t+h}^{\mathbf{u}}))] + G(x, E_{t,x} [g(t+h, X_{t+h}^{\mathbf{u}})]) + o(h). \end{aligned}$$

Combining this with the expression for  $J(t, x, \mathbf{u}_h)$  above, and the fact that (as we have proved)  $V(t, x) = J(t, x, \hat{\mathbf{u}})$ , we obtain

$$J(t, x, \hat{\mathbf{u}}) - J(t, x, \mathbf{u}_h) \geq o(h),$$

so

$$\liminf_{h \rightarrow 0} \frac{J(t, x, \hat{\mathbf{u}}) - J(t, x, \mathbf{u}_h)}{h} \geq 0,$$

and we are done. ■

## 6 The general case

We now turn to the most general case of the present paper, where the functional  $J$  is given by

$$\begin{aligned} J(t, x, \mathbf{u}) &= E_{t,x} \left[ \int_t^T H(t, x, s, X_s^{\mathbf{u}}, \mathbf{u}_s(X_s^{\mathbf{u}})) ds + F(t, x, X_T^{\mathbf{u}}) \right] \\ &\quad + G(t, x, E_{t,x} [X_T^{\mathbf{u}}]). \end{aligned} \quad (16)$$

To study the reward functional above we need a slightly modified integrability assumption.

**Assumption 6.1** *We assume that for each initial point  $(t, x) \in [0, T] \times R^n$ , and each admissible control law  $\mathbf{u}$ , we have*

$$E_{t,x} \left[ \int_t^T |H(t, x, s, X_s^{\mathbf{u}}, \mathbf{u}_s(X_s^{\mathbf{u}}))| ds + |F(x, X_T^{\mathbf{u}})| \right] < \infty, \quad (17)$$

$$E_{t,x} [|X_T^{\mathbf{u}}|] < \infty. \quad (18)$$

The treatment of this case is very similar to the previous one, so we directly give the final result, which is the relevant extended HJB system.

**Definition 6.1** *Given the objective functional (16) the extended HJB system for  $V$  is given by (19)-(24) below.*

1. *The function  $V$  is determined by*

$$\begin{aligned} \sup_{u \in R^k} \{ &(\mathbf{A}^u V)(t, x) + H(t, x, t, x, u) - (\mathbf{A}^u f)(t, x, t, x) + (\mathbf{A}^u f^{tx})(t, x) \\ &- \mathbf{A}^u (G \diamond g)(t, x) + (\mathbf{H}^u g)(t, x) \} = 0, \end{aligned} \quad (19)$$

*with boundary condition*

$$V(T, x) = F(T, x, x) + G(T, x, x). \quad (20)$$



2. For each fixed  $s$  and  $y$ , the function  $f^{sy}(t, x)$  is defined by

$$\mathbf{A}^{\hat{\mathbf{u}}} f^{sy}(t, x) + H(s, y, t, x, \hat{\mathbf{u}}_t(x)) = 0, \quad 0 \leq t \leq T \quad (21)$$

$$f^{sy}(T, x) = F(s, y, x) \quad (22)$$

3. The function  $g(t, x)$  is defined by

$$\mathbf{A}^{\hat{\mathbf{u}}} g(t, x) = 0, \quad 0 \leq t \leq T \quad (23)$$

$$g(T, x) = x. \quad (24)$$

In the definition above,  $\hat{\mathbf{u}}$  always denotes the control law which realizes the supremum in the  $V$  equation, and we have used the notation

$$\begin{aligned} f(t, x, s, y) &= f^{sy}(t, x), \\ (G \diamond g)(t, x) &= G(t, x, g(t, x)), \\ \mathbf{H}^u g(t, x) &= G_y(t, x, g(t, x)) \cdot \mathbf{A}^u g(t, x), \\ G_y(t, x, y) &= \frac{\partial G}{\partial y}(t, x, y). \end{aligned}$$

Also for this case we have a verification theorem. The proof is almost identical to that of Theorem 5.1 so we omit it.

**Theorem 6.1 (Verification Theorem)** *Assume that, for all  $(s, y)$ , the functions  $V(t, x)$ ,  $f^{sy}(t, x)$ ,  $g(t, x)$ , and  $\hat{\mathbf{u}}(t, x)$  have the following properties.*

1.  $V$ ,  $f^{sy}$ , and  $g$  is a solution to the extended HJB system in Definition 6.1.
2.  $V$ ,  $f^{sy}$ , and  $g$  are smooth in the sense that they are in  $C^{1,2}$ .
3. The function  $\hat{\mathbf{u}}$  realizes the supremum in the  $V$  equation, and  $\hat{\mathbf{u}}$  is an admissible control law.
4.  $V$ ,  $f^{sy}$ ,  $g$ , and  $G \diamond g$ , as well as the function  $(t, x) \mapsto f(t, x, t, x)$  all belong to the space  $L^2(X^{\hat{\mathbf{u}}})$ .

Then  $\hat{\mathbf{u}}$  is an equilibrium law, and  $V$  is the corresponding equilibrium value function. Furthermore,  $f$ , and  $g$  have the probabilistic representations

$$f^{sy}(t, x) = E_{t,x} \left[ \int_t^T H(s, y, r, X_r^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_r(X_r^{\hat{\mathbf{u}}})) dr + F(s, y, X_T^{\hat{\mathbf{u}}}) \right], \quad (25)$$

$$g(t, x) = E_{t,x} [X_T^{\hat{\mathbf{u}}}], \quad 0 \leq t \leq T. \quad (26)$$

## 7 Special cases and extensions

In this section we comment on possible extensions and a couple of important special cases.

## 7.1 A driving point process

In the present paper we have, for notational clearness, confined ourselves to a pure diffusion framework. It is, however, very easy to extend the theory to a case where the SDE, apart from the Wiener process, is also driven by a marked point process with Markovian characteristics. The extended HJB system will look exactly the same as above, but the form of the infinitesimal operator  $\mathbf{A}^u$  will of course change, and we would need to slightly modify the integrability assumptions.

## 7.2 The case when $G = 0$ .

In the case when the term  $G$  is not present, the  $V$  equation takes the form

$$\sup_{u \in R^k} \{(\mathbf{A}^u V)(t, x) + H(t, x, t, x, u) - (\mathbf{A}^u f)(t, x, t, x) + (\mathbf{A}^u f^{tx})(t, x)\} = 0$$

In this case, however, it follows from the probabilistic representation of  $f$  that  $f(t, x, t, x) = V(t, x)$  so we have a cancellation in the  $V$ -equation. The HJB system (19)-(24) is thus replaced by the much simpler system

$$\sup_{u \in R^k} \{H(t, x, t, x, u) + (\mathbf{A}^u f^{tx})(t, x)\} = 0, \quad (27)$$

$$\mathbf{A}^{\hat{u}} f^{sy}(t, x) + H(s, y, t, x, \hat{u}_t(x)) = 0, \quad (28)$$

$$f^{sy}(T, x) = F(s, y, x). \quad (29)$$

## 7.3 Infinite horizon

The results above can easily be extended to the case with infinite horizon, i.e. when  $T = +\infty$ . The natural reward functional will then have the form

$$J(t, x, \mathbf{u}) = E_{t,x} \left[ \int_t^\infty H(t, x, s, X_s^{\mathbf{u}}, \mathbf{u}_s(X_s^{\mathbf{u}})) ds \right]$$

so the functions  $F$  and  $G$  are not present. It is easy to see that for this case we have the extended HJB system

$$\sup_{u \in R^k} \{(\mathbf{A}^u V)(t, x) + H(t, x, t, x, u) - (\mathbf{A}^u f)(t, x, t, x) + (\mathbf{A}^u f^{tx})(t, x)\} = 0,$$

$$\lim_{T \rightarrow \infty} E_{t,x} [V(T, X_T^{\hat{u}})] = 0,$$

$$\mathbf{A}^{\hat{u}} f^{sy}(t, x) + H(s, y, t, x, \hat{u}_t(x)) = 0,$$

$$\lim_{T \rightarrow \infty} E_{t,x} [f^{sy}(T, X_T^{\hat{u}})] = 0$$

We also have a verification theorem where the proof is almost identical to the earlier case.

## 7.4 Generalizing $H$ and $G$

We can easily extend the result above to the case when the term  $G(t, x, E_{t,x}[X_T^{\mathbf{u}}])$  is replaced by

$$G(t, x, E_{t,x}[k(X_T^{\mathbf{u}})])$$

for some function  $k$ . In this case we simply define  $g$  by

$$g(t, x) = E_{t,x}[k(X_T^{\hat{\mathbf{u}}})].$$

The extended HJB system then looks exactly as in Definition 6.1 above, apart from the fact that the boundary condition for  $g$  is changed to

$$g(T, x) = k(x).$$

See [14] for an interesting application.

It is also possible to extend the  $H$  term to be of the form

$$H(t, x, s, X_s, E_{t,x}[b(X_s^{\mathbf{u}})], u_s)$$

in the integral term of the value functional. The structure of the resulting HJB system is fairly obvious but we have omitted it since the present HJB system is, in our opinion, complicated enough as it is.

## 7.5 The case with no state dependence

We see that the general extended HJB equation is quite complicated. In many concrete cases there are, however, cancellations between different terms in the equation. The simplest case occurs when the objective functional has the form

$$J(t, x, \mathbf{u}) = E_{t,x}[F(X_T^{\mathbf{u}})] + G(E_{t,x}[X_T^{\mathbf{u}}]),$$

so  $F$  and  $G$  do not depend on the present state  $x$ , and  $X$  is a scalar diffusion of the form

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t.$$

In this case the extended HJB equation has the form

$$\sup_{u \in R^k} \{ \mathbf{A}^u V(t, x) - \mathbf{A}^u [G(g(t, x))] + G'(g(t, x)) \mathbf{A}^u g(t, x) \} = 0,$$

and a simple calculation shows that

$$-\mathbf{A}^u [G(g(t, x))] + G'(g(t, x)) \mathbf{A}^u g(t, x) = -\frac{1}{2} \sigma^2(x, u) G''(g(t, x)) g_x^2(t, x),$$

where  $g_x = \frac{\partial g}{\partial x}$ . Thus the extended HJB equation becomes

$$\sup_{u \in R^k} \left\{ \mathbf{A}^u V(t, x) - \frac{1}{2} \sigma^2(x, u) G''(g(t, x)) g_x^2(t, x) \right\} = 0, \quad (30)$$

## 7.6 A scaling result

In this section we derive a small scaling result. Let us thus consider the objective functional (16) above and denote, as usual, the equilibrium control and value function by  $\hat{\mathbf{u}}$  and  $V$  respectively. Let  $\varphi : R^n \rightarrow R$  be a fixed real valued function and consider a new objective functional  $J_\varphi$ , defined by,

$$J_\varphi(t, x, \mathbf{u}) = \varphi(x)J(t, x, \mathbf{u}), \quad n = 0, 1, \dots, T$$

and denote the corresponding equilibrium control and value function by  $\hat{\mathbf{u}}_\varphi$  and  $V_\varphi$  respectively. Since Player  $t$  is (loosely speaking) trying to maximize  $J_\varphi(t, x, \mathbf{u})$  over  $u_t$ , and  $\varphi(x)$  is just a scaling factor which is not affected by  $u_t$  the following result is intuitively obvious. The formal proof is, however, not quite trivial.

**Proposition 7.1** *With notation as above we have*

$$\begin{aligned} V_\varphi(t, x) &= \varphi(x)V(t, x), \\ \hat{\mathbf{u}}_\varphi(t, x) &= \hat{\mathbf{u}}(t, x). \end{aligned}$$

**Proof.** For notational simplicity we consider the case when  $J$  is of the form

$$J(t, x, \mathbf{u}) = E_{t,x} [F(x, X_T^{\mathbf{u}})] + G(x, E_{t,x} [X_T^{\mathbf{u}}]). \quad (31)$$

The proof for the general case has exactly the same structure.

For  $J$  as above we have the extended HJB system

$$\begin{aligned} \sup_{u \in R^k} \{ & (\mathbf{A}^u V)(t, x) - (\mathbf{A}^u f)(t, x, x) + (\mathbf{A}^u f^x)(t, x) \\ & - \mathbf{A}^u (G \diamond g)(t, x) + G_y(x, g(t, x)) \cdot \mathbf{A}^u g(t, x) \} &= 0, \quad 0 \leq t \leq T, \\ & \mathbf{A}^{\hat{\mathbf{u}}} f^y(t, x) &= 0, \quad 0 \leq t \leq T, \\ & \mathbf{A}^{\hat{\mathbf{u}}} g(t, x) &= 0, \quad 0 \leq t \leq T, \\ & V(T, x) &= F(x, x) + G(x, x), \\ & f^y(T, x) &= F(y, x), \\ & g(T, x) &= x. \end{aligned}$$

We now recall the probabilistic interpretations

$$\begin{aligned} V(t, x) &= E_{t,x} [F(x, X_T^{\hat{\mathbf{u}}})] + G(x, E_{t,x} [X_T^{\hat{\mathbf{u}}})] \\ f(t, x, y) &= E_{t,x} [F(y, X_T^{\hat{\mathbf{u}}})], \\ g(t, x) &= E_{t,x} [X_T^{\hat{\mathbf{u}}}). \end{aligned}$$

and the definition

$$(G \diamond g)(t, x) = G(x, g(t, x)).$$

From this it follows that

$$V(t, x) = f(t, x, x) + (G \diamond g)(t, x),$$

so the first HJB equation above can be written

$$\sup_{u \in R^k} \{(\mathbf{A}^u f^x)(t, x) + G_y(x, g(t, x)) \cdot \mathbf{A}^u g(t, x)\} = 0.$$

We now turn to  $J_\varphi$ , which can be written

$$J(t, x, \mathbf{u}) = E_{t,x} [F_\varphi(x, X_T^\mathbf{u})] + G_\varphi(x, E_{t,x} [X_T^\mathbf{u}]),$$

where

$$\begin{aligned} F_\varphi(x, y) &= \varphi(x)F(x, y), \\ G_\varphi(x, y) &= \varphi(x)G(x, y), \end{aligned}$$

and we note that

$$\frac{\partial G_\varphi}{\partial y}(x, y) = \varphi(x)G_y(x, y)$$

We thus obtain the HJB equation

$$\sup_{u \in R^k} \{(\mathbf{A}^u f_\varphi^x)(t, x) + \varphi(x)G_y(x, g_\varphi(t, x)) \cdot \mathbf{A}^u g_\varphi(t, x)\} = 0,$$

with  $f_\varphi$  and  $g_\varphi$  defined by

$$\begin{aligned} \mathbf{A}^{\hat{u}_\varphi} f_\varphi^y(t, x) &= 0, \\ \mathbf{A}^{\hat{u}_\varphi} g_\varphi(t, x) &= 0, \\ f_\varphi^y(T, x) &= \varphi(y)F(y, x), \\ g_\varphi(T, x) &= x. \end{aligned}$$

From this it follows that we can write

$$\begin{aligned} f_\varphi(t, x, y) &= \varphi(y)f_0(t, x, y), \\ g_\varphi(t, x) &= g_0(t, x) \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}^{\hat{u}_\varphi} f_0^y(t, x) &= 0, \\ \mathbf{A}^{\hat{u}_\varphi} g_0(t, x) &= 0, \\ f_0^y(T, x) &= F(y, x), \\ g_0(T, x) &= x. \end{aligned}$$

and the HJB equation has the form

$$\sup_{u \in R^k} \{\varphi(x)(\mathbf{A}^u f_0^x)(t, x) + \varphi(x)G_y(x, g_0(t, x)) \cdot \mathbf{A}^u g_0(t, x)\} = 0,$$

or, equivalently,

$$\sup_{u \in R^k} \{(\mathbf{A}^u f_0^x)(t, x) + G_y(x, g_0(t, x)) \cdot \mathbf{A}^u g_0(t, x)\} = 0.$$

We thus see that the system for  $f$ ,  $g$ , and  $\hat{\mathbf{u}}$  is exactly the same as that for  $f_0$ ,  $g_0$ , and  $\hat{\mathbf{u}}_\varphi$ . We thus have

$$\begin{aligned} f_\varphi(t, x, y) &= \varphi(y)f(t, x, y), \\ g_\varphi(t, x) &= g(t, x), \\ \hat{\mathbf{u}}_\varphi &= \hat{\mathbf{u}}. \end{aligned}$$

Moreover, since

$$V_\varphi(t, x) = f_\varphi(t, x, x) + (G_\varphi \diamond g_\varphi)(t, x),$$

we obtain

$$V_\varphi(t, x) = \varphi(x)V(t, x).$$

■

## 8 An equivalent time consistent problem

The object of the present section is to provide a link between time inconsistent and time consistent problems. To this end we go back to the general continuous time extended HJB system (19)-(24). The  $V$ -equation (19) reads as

$$\begin{aligned} \sup_{u \in R^k} \{ (\mathbf{A}^u V)(t, x) + H(t, x, t, x, u) - (\mathbf{A}^u f)(t, x, t, x) + (\mathbf{A}^u f^{tx})(t, x) \\ - \mathbf{A}^u (G \diamond g)(t, x) + (\mathbf{H}^u g)(t, x) \} = 0, \end{aligned}$$

Let us now assume that there exists an equilibrium control law  $\hat{\mathbf{u}}$ . Using  $\hat{\mathbf{u}}$  we can then construct  $f$  and  $g$  by solving the associated equations (21)-(24). We now define the function  $K$  by

$$\begin{aligned} K(t, x, u) &= H(t, x, t, x, u) - (\mathbf{A}^u f)(t, x, t, x) + (\mathbf{A}^u f^{tx})(t, x) \\ &\quad - \mathbf{A}^u (G \diamond g)(t, x) + (\mathbf{H}^u g)(t, x). \end{aligned}$$

With this definition of  $K$ , the equation for  $V$  above and its boundary condition become

$$\begin{aligned} \sup_{u \in R^k} \{ (\mathbf{A}^u V)(t, x) + K(t, x, u) \} &= 0, \\ V(T, x) &= F(x, x) + G(x, x). \end{aligned}$$

We now observe, by inspection, that this is a standard HJB equation for the standard time consistent optimal control problem to maximize

$$E_{t,x} \left[ \int_t^T K(s, X_s, u_s) ds + F(X_T, X_T) + G(X_T, X_T) \right]. \quad (32)$$

We have thus proved the following result.

**Proposition 8.1** *For every time inconsistent problem in the present framework there exists a standard, time consistent optimal control problem with the following properties.*

- *The optimal value function for the standard problem coincides with the equilibrium value function for the time inconsistent problem.*
- *The optimal control for the standard problem coincides with the equilibrium control for the time inconsistent problem.*
- *The objective functional for the standard problem is given by (32).*

We immediately remark that the Proposition above is mostly of theoretical interest, and of little “practical” value. The reason is of course that in order to formulate the equivalent standard problem we need to know the equilibrium control  $\hat{\mathbf{u}}$ .

Related results can be found in [1], [7], [10] and [15]. In these papers it is proved that, for various models where time inconsistency stems from non-exponential discounting, there exists an equivalent standard problem (with exponential discounting).

Proposition 8.1 differs from the results in the cited references above two ways. Firstly it differs by being quite general and not confined to a particular model. Secondly it differs from the results in the cited references by having a different structure. In other words, for the models studied in the cited papers, the equivalent problem described in Proposition 8.1 is structurally different from the equivalent problems presented in the cited references. See Section 8.3 of [4] for a more detailed discussion of issues of this kind.

## 9 Example: Non exponential discounting

We now illustrate the theory developed above, and the first example we consider is a fairly general case of a control problem with non exponential discounting. This topic has previously been treated in [16] with a slightly different formalism. We thus have no claim to originality, and the purpose of this section is merely to see how the special case of non exponential discounting falls into the general theory developed above.

### 9.1 The general case

Our general model is specified as follows.

- We consider the same controlled SDE as before.
- The reward functional for Player  $t$  is given by

$$J(t, x, \mathbf{u}) = E_{tx} \left[ \int_t^T \beta(s-t) H(X_s^{\mathbf{u}}, \mathbf{u}_s(X_s^{\mathbf{u}})) ds + \beta(T-t) F(X_T^{\mathbf{u}}) \right],$$

where the discounting function  $\beta(t)$ , the local utility function  $H(x, u)$  and the final state utility function  $F(x)$  are deterministic functions.

- We assume that the discounting function  $\beta$  is non-negative and integrable over  $[0, \infty)$ . Without loss of generality we assume that

$$\beta(0) = 1.$$

In the notation of Definition 6.1 we see that we have no  $G$ -term, and we have no dependence on the initial state  $x$  in the utility functions  $H$  and  $F$ . The function  $f^{sy}(t, x)$  will thus be of the form  $f^s(t, x)$  and the extended HJB equation takes the form

$$\sup_{u \in R^k} \{(\mathbf{A}^u V)(t, x) + H(x, u) - (\mathbf{A}^u f)(t, x, t) + (\mathbf{A}^u f^t)(t, x)\} = 0,$$

where we have used the notation  $f(t, x, s) = f^s(t, x)$ . Since  $\mathbf{A}^u$  operates on variables in parenthesis, we obtain

$$-(\mathbf{A}^u f)(t, x, t) + (\mathbf{A}^u f^t)(t, x) = -\frac{\partial f}{\partial s}(t, x, t),$$

where the partial derivative  $\frac{\partial f}{\partial s}$  acts on the last  $t$  in  $f(t, x, t)$ . We thus see that the extended HJB system has the form

$$\begin{aligned} \sup_{u \in R^k} \left\{ \mathbf{A}^u V(t, x) + H(x, u) - \frac{\partial f}{\partial s}(t, x, t) \right\} &= 0, \\ V(T, x) &= F(x), \end{aligned}$$

where  $f^s$  is determined by

$$\begin{aligned} \mathbf{A}^{\hat{u}} f^s(t, x) + \beta(t-s)H(x, \hat{u}_t(x)) &= 0, \quad s \leq t \leq T, \\ f^s(T, x) &= \beta(T-s)F(x), \end{aligned}$$

with probabilistic interpretation

$$f^s(t, x) = E_{t,x} \left[ \int_t^T \beta(r-s)H(X_r^{\hat{u}}, \hat{u}_r(X_r^{\hat{u}})) dr + \beta(T-s)F(X_T^{\hat{u}}) \right].$$

We may now define the function  $g^s$  by

$$g^s(t, x) = -\frac{\partial f}{\partial s}(t, x, s).$$

Taking the  $s$ -derivative in the representation for  $f^s$  above gives us our HJB system.



**Proposition 9.1** *The extended HJB system has the form*

$$\begin{aligned} \sup_{u \in R^k} \{ \mathbf{A}^u V(t, x) + H(x, u) + g(t, x, t) \} &= 0, \\ V(T, x) &= F(x), \end{aligned}$$

where  $g^s(t, x) = g(t, x, s)$  is determined by

$$\begin{aligned} \mathbf{A}^{\hat{\mathbf{u}}} g^s(t, x) + \beta'(t-s)H(x, \hat{\mathbf{u}}_t(x)) &= 0, \quad s \leq t \leq T, \\ g^s(T, x) &= \beta'(T-s)F(x), \end{aligned}$$

with probabilistic interpretation

$$g^s(t, x) = E_{t,x} \left[ \int_t^T \beta'(r-s)H(X_r^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}_r(X_r^{\hat{\mathbf{u}}})) dr + \beta'(T-s)F(X_T^{\hat{\mathbf{u}}}) \right].$$

This generalizes the corresponding results in [7] and [8], where special cases are treated in great detail.

## 9.2 Infinite horizon

We now move to the case of infinite horizon and we restrict ourselves to the time invariant case. We thus assume that the reward functional is of the form

$$J(t, x, \mathbf{u}) = E_{t,x} \left[ \int_t^\infty \beta(s-t)H(X_s^{\mathbf{u}}, \mathbf{u}_s(X_s^{\mathbf{u}})) ds \right],$$

and that the  $X$  dynamics are of the form

$$dX_t = \mu(X_t, u_t)dt + \sigma(X_t, u_t)dW_t.$$

For this case it is natural to look for a time invariant solution, i.e. to study the case when  $V$  is of the form  $V(t, x) = V(0, x) = V(x)$ , the control  $\mathbf{u}$  is of the form  $\mathbf{u}(t, x) = \mathbf{u}(0, x) = \mathbf{u}(x)$ , and  $g$  is of the form  $g^s(t, x) = g^0(t-s, x)$ . We now define the function  $h$  by

$$h(t, x) = g^0(t, x),$$

and, after some elementary calculations, we have the following result.

**Proposition 9.2** *For the time invariant case with infinite horizon, the extended HJB system has the form*

$$\sup_{u \in R^k} \{ \mathbf{A}^u V(x) + H(x, u) + h(0, x) \} = 0,$$

where  $h$  is determined by

$$\begin{aligned} \mathbf{A}^{\hat{\mathbf{u}}} h(t, x) + \beta'(t)H(x, \hat{\mathbf{u}}(x)) &= 0, \quad t \geq 0 \\ \lim_{t \rightarrow \infty} E_{t,x} [h(t, X_t^{\hat{\mathbf{u}}})] &= 0, \end{aligned}$$

with probabilistic interpretation

$$h(t, x) = E_{t,x} \left[ \int_t^\infty \beta'(s) H(X_s^{\hat{u}}, \hat{u}(X_s^{\hat{u}})) ds \right]. \quad t \geq 0$$

so we can also write the extended HJB eqn as

$$\sup_{u \in R^k} \left\{ \mathbf{A}^u V(x) + H(x, u) + E_{0,x} \left[ \int_0^\infty \beta'(s) H(X_s^{\hat{u}}, \hat{u}(X_s^{\hat{u}})) ds \right] \right\} = 0,$$

**Remark 9.1** Since  $V$  is not a function of  $t$ , the  $\frac{\partial}{\partial t}$ -term in the operator  $\mathbf{A}^u$  will not come into action in the  $V$ -equation.

**Remark 9.2** In the time consistent case where we have the exponential discounting  $\beta(t) = e^{-\delta t}$  we have

$$h(0, x) = -\delta E_{0,x} \left[ \int_0^\infty e^{-\delta s} H(X_s^{\hat{u}}, \hat{u}(X_s^{\hat{u}})) ds \right] = -\delta V(x),$$

so we have the standard HJB equation

$$\sup_{u \in R^k} \{ \mathbf{A}^u V(x) + H(x, u) \} = \delta V(x).$$

## 10 Example: The inconsistent linear quadratic regulator

To illustrate how the theory works in a simple case, we consider a small variation of the classical linear quadratic regulator. The model is specified as follows.

- The value functional for Player  $t$  is given by

$$E_{t,x} \left[ \frac{1}{2} \int_t^T u_s^2 ds \right] + \frac{\gamma}{2} E_{t,x} \left[ (X_T - x)^2 \right]$$

where  $\gamma$  is a positive constant.

- The state process  $X$  is scalar with dynamics

$$dX_t = [aX_t + bu_t] dt + \sigma dW_t,$$

where  $a$ ,  $b$  and  $\sigma$  are given constants.

- The control  $u$  is scalar with no constraints.

This is a time inconsistent version of the classical linear quadratic regulator. Loosely speaking we want control the system so that the final state  $X_T$  is close to  $x$  while at the same time keeping the control energy (formalized by the integral term) is small. The time inconsistency stems from the fact that the

target point  $x = X_t$  is changing as time goes by. In discrete time this problem is studied in [4].

For this problem we have

$$\begin{aligned} F(x, y) &= \frac{\gamma}{2}(y - x)^2, \\ H(u) &= \frac{1}{2}u^2 \end{aligned}$$

and as usual we introduce the functions  $f^y(t, x)$  and  $f(t, x, x)$  by

$$\begin{aligned} f^y(t, x) &= E_{t,x} \left[ \int_t^T \frac{1}{2} \hat{\mathbf{u}}_s^2 (X_s^{\hat{\mathbf{u}}}) ds + (X_T^{\hat{\mathbf{u}}} - y)^2 \right], \\ f(t, x, y) &= f^y(t, x). \end{aligned}$$

The extended HJB system takes the form

$$\begin{aligned} \inf_u \left\{ \frac{1}{2}u^2 + \mathbf{A}^u f^x(t, x) \right\} &= 0, \quad 0 \leq t \leq T \\ \mathbf{A}^{\hat{\mathbf{u}}} f^y(t, x) + \frac{1}{2} \hat{\mathbf{u}}_t^2(x) &= 0, \quad 0 \leq t \leq T \\ f^y(T, x) &= \frac{\gamma}{2}(x - y)^2. \end{aligned}$$

From the  $X$  dynamics we see that

$$\mathbf{A}^u = \frac{\partial}{\partial t} + (ax + bu) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2}.$$

We thus obtain the following form of the HJB equation, where for shortness of notation we denote partial derivatives by lower case index so, for example,  $f_x = \frac{\partial f}{\partial x}$ .

$$\begin{aligned} \inf_u \left\{ \frac{1}{2}u^2 + f_t(t, x, x) + (ax + bu) f_x(t, x, x) + \frac{1}{2} \sigma^2 f_{xx}(t, x, x) \right\} &= 0, \\ f(T, x, x) &= 0. \end{aligned}$$

The coupled system for  $f^y$  is given by

$$\begin{aligned} f_t^y(t, x) + [ax + b\hat{\mathbf{u}}(t, x)] f_x^y(t, x) + \frac{1}{2} \sigma^2 f_{xx}^y(t, x) + \frac{1}{2} \hat{\mathbf{u}}^2(t, x) &= 0, \\ f^y(T, x) &= \frac{\gamma}{2}(x - y)^2. \end{aligned}$$

The first order condition in the HJB equation gives us

$$\hat{\mathbf{u}}(t, x) = -bf_x(t, x, x)$$

and, inspired of the standard regulator problem, we now make the Ansatz (attempted solution)

$$f(t, x, y) = A(t)x^2 + B(t)y^2 + C(t)xy + D(t)x + F(t)y + H(t), \quad (33)$$

where all coefficients are deterministic functions of time. We now insert the Ansatz into the HJB system, and perform a number of extremely boring calculations. As a result of these calculations, it turns out that the variables separate in the expected way and we have the following result.

**Proposition 10.1** *For the time inconsistent regulator, we have the structure (33), where the coefficient functions solve the following system of ODEs.*

$$\begin{aligned}
A_t + 2aA - 2b^2A(2A + C) + \frac{1}{2}b^2(2A + C)^2 &= 0 \\
B_t &= 0 \\
C_t + aC - b^2C(2A + C) &= 0 \\
D_t + aD - 2b^2AD &= 0 \\
F_t - b^2CD &= 0 \\
H_t - \frac{1}{2}b^2D^2 + \sigma^2A &= 0
\end{aligned}$$

*With boundary conditions*

$$\begin{aligned}
A(T) = \frac{\gamma}{2}, \quad B(T) &= \frac{\gamma}{2}, \quad C(T) = -\gamma, \\
D(T) = 0, \quad F(T) &= 0, \quad H(T) = 0.
\end{aligned}$$

## 11 Example: A Cox-Ingersoll-Ross production economy with time inconsistent preferences

In this section we apply the previously developed theory to a rather detailed study of a general equilibrium model for a production economy with time inconsistent preferences. The model under consideration is a time inconsistent analogue of the classic Cox-Ingersoll-Ross model in [5]. Our main objective is to investigate the structure of the equilibrium short rate, the equilibrium Girsanov kernel, and the equilibrium stochastic discount factor.

There are a few earlier papers on equilibrium with time inconsistent preferences. In [1] and [15] the authors study continuous time equilibrium models of a particular type of time inconsistency, namely non-exponential discounting. While [1] considers a deterministic neoclassical model of economic growth, [15] analyze general equilibrium in a stochastic endowment economy.

Our present study is much inspired by the earlier paper [12] which, in very great detail, studies equilibrium in a very general setting of an endowment economy with dynamically inconsistent preferences that is not limited to the particular case of non-exponential discounting.

Unlike the papers mentioned above, which all studies *endowment* models, we study a stochastic *production* economy of Cox-Ingersoll-Ross type.

### 11.1 The Model

We start with some formal assumptions concerning the production technology.

**Assumption 11.1** *We assume that there exists a constant returns to scale physical production technology process  $S$  with dynamics*

$$dS_t = \alpha S_t dt + S_t \sigma dW_t. \quad (34)$$

*The economic agents can invest unlimited positive amounts in this technology, but since it is a matter of physical investment, short positions are not allowed.*

More concretely this means that at any time you are allowed to invest dollars in the production process. If you, at time  $t$ , invest  $q$  dollars, and wait until time  $u$  then you will receive the amount of  $q \cdot S_u / S_t$  in dollars. In particular we see that the return on the investment is linear in  $q$ , hence the term “constant returns to scale”. Since this is a matter of physical investment, shortselling is not allowed.

A moment of reflection shows that, from a purely formal point of view, investment in the technology  $S$  is in fact equivalent to the possibility of investing in a risky asset with price process  $S$ , but again with the constraint that shortselling is not allowed.

We also need a risk free asset, and this is provided by the next assumption.

**Assumption 11.2** *We assume that there exists a risk free asset in zero net supply with dynamics*

$$dB_t = r_t B_t dt,$$

*where  $r$  is the short rate process, which will be determined endogenously. The risk free rate  $r$  is assumed to be of the form*

$$r_t = r(t, X_t)$$

*where  $X$  denotes the portfolio value of the representative investor (to be defined below).*

Interpreting the production technology  $S$  as above, the wealth dynamics will be given by

$$dX_t = X_t u_t (\alpha - r_t) dt + (r_t X_t - c_t) dt + X_t u_t \sigma dW_t.$$

where  $u$  is the portfolio weight on production, so  $1 - u$  is the weight on the risk free asset. Finally we need an economic agent.

**Assumption 11.3** *We assume that there exists a representative agent who, at every point  $(t, x)$ , wishes to maximize the reward functional*

$$E_{t,x} \left[ \int_t^T U(t, x, s, c_s) ds \right]. \quad (35)$$

## 11.2 Equilibrium definitions

We now go on to study equilibrium in our model. We will in fact have two equilibrium concepts

- Intrapersonal equilibrium.
- Market equilibrium.

The intrapersonal equilibrium is related to the lack of time consistency in preferences, whereas the market equilibrium is related to market clearing. We now discuss these concepts in more detail.

### 11.2.1 Intrapersonal equilibrium

Consider, for a given short rate function  $r(t, x)$  the control problem with reward functional

$$E_{t,x} \left[ \int_t^T U(t, x, s, c_s) ds \right].$$

and wealth dynamics

$$dX_t = X_t u_t (\alpha - r_t) dt + (r_t X_t - c_t) dt + X_t u_t \sigma dW_t.$$

where  $r_t$  is shorthand for  $r(t, X_t)$ . If the agent wants to maximize the reward functional for every initial point  $(t, x)$  then, because of the appearance of  $(t, x)$  in the utility function  $U$ , this is a time inconsistent control problem. In order to handle this situation we use the game theoretic setup and results developed in Sections 1-6 above. This subgame perfect Nash equilibrium concept is henceforth referred to as the **intrapersonal equilibrium**.

### 11.2.2 Market equilibrium

By a **market equilibrium** we mean a situation where the agent follows an intrapersonal equilibrium strategy, and where the market clears for the risk free asset. The formal definition is as follows.

**Definition 11.1** *A market equilibrium of the model is a triple of real valued functions  $\{\hat{c}(t, x), \hat{u}(t, x), r(t, x)\}$  such that the following hold.*

1. *Given the risk free rate process  $r(t, x)$ , the intrapersonal equilibrium consumption and investment are given by  $\hat{c}$  and  $\hat{u}$  respectively.*
2. *The market clears for the risk free asset, i.e.*

$$\hat{u}(t, x) \equiv 1.$$

### 11.3 Main goals of the study

As will be seen below, there will be a unique equilibrium martingale measure  $Q$  with corresponding likelihood process  $L = dQ/dP$ , where  $L$  has dynamics

$$dL_t = L_t \varphi_t dW_t.$$

The process  $\varphi$  will be referred to as the equilibrium *Girsanov kernel*. There will also be an equilibrium short rate process  $r$ , which will be related to  $\varphi$  by the standard no arbitrage relation

$$r(t, x) = \alpha + \varphi(t, x)\sigma, \quad (36)$$

which says that  $S/B$  is a  $Q$ -martingale. There will also be a unique equilibrium stochastic discount factor  $M$  defined by

$$M_t = e^{-\int_0^t r_s ds} L_t.$$

For ease of notation we will, however, only identify the stochastic discount factor  $M$ , up to a multiplicative constant, so for any arbitrage free (non dividend) price process  $p_t$  we will have the pricing equation

$$p_s = \frac{1}{M_s} E^P [M_t p_t | \mathcal{F}_s].$$

Our goal is to obtain expressions for  $\varphi$ ,  $r$  and  $M$ .

### 11.4 The extended HJB equation

In order to determine the intrapersonal equilibrium we use the results from Section 7.2. The extended HJB equation (27) now reads as

$$\sup_{u \geq 0, c \geq 0} \{U(t, x, t, c) + \mathbf{A}^{u,c} f^{tx}(t, x)\} = 0 \quad (37)$$

and  $f^{sy}$  is determined by

$$\mathbf{A}^{\hat{u}, \hat{c}} f^{sy}(t, x) + U(s, y, t, \hat{c}(t, x)) = 0 \quad (38)$$

with the probabilistic representation

$$f^{sy}(t, x) = E_{t,x} \left[ \int_t^T U(s, y, \tau, \hat{c}(\tau, X_\tau^{\hat{u}})) d\tau \right], \quad 0 \leq t \leq T. \quad (39)$$

The term  $\mathbf{A}^{u,c} f^{tx}(t, x)$  is given by

$$\mathbf{A}^{u,c} f^{tx}(t, x) = f_t + xu(\alpha - r)f_x + (rx - c)f_x + \frac{1}{2}x^2u^2\sigma^2 f_{xx} \quad (40)$$

where  $f$  and the derivatives are evaluated at  $(t, x, t, x)$  and where we have used the notation

$$\begin{aligned} f(t, x, s, y) &= f^{sy}(t, x), \\ f_t(t, x, s, y) &= \frac{\partial f}{\partial t}(t, x, s, y), \\ f_x(t, x, s, y) &= \frac{\partial f}{\partial x}(t, x, s, y), \\ f_{xx}(t, x, s, y) &= \frac{\partial^2 f}{\partial x^2}(t, x, s, y). \end{aligned}$$

The first order conditions for an interior optimum are

$$U_c(t, x, t, \hat{c}) = f_x(t, x, t, x) \quad (41)$$

$$\hat{u} = - \left( \frac{\alpha - r}{\sigma^2} \right) \frac{f_x(t, x, t, x)}{x f_{xx}(t, x, t, x)}. \quad (42)$$

## 11.5 Determining market equilibrium

In order to determine the market equilibrium we use the equilibrium condition  $\hat{u} = 1$ . Plugging this into (42) we immediately obtain our first result.

**Proposition 11.1** *With assumptions as above the following hold.*

- The equilibrium short rate is given by

$$r(t, x) = \alpha + \sigma^2 \frac{x f_{xx}(t, x, t, x)}{f_x(t, x, t, x)}. \quad (43)$$

- The equilibrium Girsanov kernel  $\varphi$  is given by

$$\varphi(t, x) = \sigma \frac{x f_{xx}(t, x, t, x)}{f_x(t, x, t, x)}. \quad (44)$$

- The extended equilibrium HJB system has the form

$$U(t, x, t, \hat{c}) + f_t + (\alpha x - \hat{c}) f_x + \frac{1}{2} x^2 \sigma^2 f_{xx} = 0, \quad (45)$$

$$\mathbf{A}^{\hat{c}} f^{sy}(t, x) + U(s, y, t, \hat{c}(t, x)) = 0 \quad (46)$$

- The equilibrium consumption  $\hat{c}$  is determined by the first order condition

$$U_c(t, x, t, \hat{c}) = f_x(t, x, t, x) \quad (47)$$

- The term  $\mathbf{A}^{\hat{c}} f^{tx}(t, x)$  is given by

$$\mathbf{A}^{\hat{c}} f^{tx}(t, x) = f_t + x(\alpha - r) f_x + (rx - \hat{c}) f_x + \frac{1}{2} x^2 \sigma^2 f_{xx} \quad (48)$$



- The equilibrium  $X$  dynamics are given by

$$dX_t = (\alpha X_t - \hat{c}_t) dt + X_t \sigma dW_t. \quad (49)$$

**Proof.** The formula (44) follows from (43) and (36). The other results are obvious. ■

## 11.6 Recap of standard results

We can compare the results above with the standard case where the utility functional for the agent is of the time consistent form

$$E_{t,x} \left[ \int_t^T U(s, c_s) ds \right].$$

In this case we have a standard HJB equation of the form

$$\sup_{u \in R, c \geq 0} \{U(t, c) + \mathbf{A}^{u,c} V(t, x)\} = 0. \quad (50)$$

and the equilibrium quantities are given by the well known expressions

$$r(t, x) = \alpha + \sigma^2 \frac{x V_{xx}(t, x)}{V_x(t, x)}, \quad (51)$$

$$\varphi(t, x) = \sigma \frac{x V_{xx}(t, x)}{V_x(t, x)}. \quad (52)$$

We note the strong structural similarities between the old and the new formulas, but we also note important differences. Let us take the formulas for the equilibrium short rate  $r$  as an example. We recall the standard and time inconsistent formulas

$$r(t, x) = \alpha + \sigma^2 \frac{x V_{xx}(t, x)}{V_x(t, x)}, \quad (53)$$

$$r(t, x) = \alpha + \sigma^2 \frac{x f_{xx}(t, x, t, x)}{f_x(t, x, t, x)}. \quad (54)$$

For the time inconsistent case we do have the relation  $V^e(t, x) = f(t, x, t, x)$  (where temporarily, and for the sake of clearness,  $V^e$  denotes the equilibrium value function) so it is tempting to think that we should be able to write (54) as

$$r(t, x) = \alpha + \sigma^2 \frac{x V_{xx}^e(t, x)}{V_x^e(t, x)}$$

which would be structurally identical to (53). Not however, that while we do have  $f(t, x, t, x) = V^e(t, x)$ , the partial derivative  $f_x(t, x, t, x)$  is **not** equal to  $V_x^e(t, x)$ . The reason is that in  $f_x(t, x, t, x)$ , the partial differential operator only acts on the first occurrence of  $x$  in  $f(t, x, t, x)$ , whereas  $V_x^e(t, x)$  will be the total  $x$ -derivative of  $f(t, x, t, x)$ .

## 11.7 The stochastic discount factor

We now go on to investigate our main object of interest, namely the equilibrium stochastic discount factor  $M$ . We recall from general arbitrage theory that

$$M_t = e^{-\int_0^t r_u du} L_t \quad (55)$$

where  $L$  is the likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

with dynamics

$$dL_t = L_t \varphi_t dW_t.$$

From this we immediately obtain the  $M$  dynamics as

$$dM_t = -r_t M_t dt + M_t \varphi_t dW_t, \quad (56)$$

so we can identify the short rate  $r$  and the Girsanov kernel  $\varphi$  from the dynamics of  $M$ .

From Proposition 11.1 we know  $r$  and  $\varphi$ , so in principle we have in fact already determined  $M$ , but we now want to investigate the relation between  $M$ , the direct utility function  $U$ , and the indirect utility function  $f$  in the extended HJB equation.

We recall from standard theory that for the usual time consistent case the (non normalized) stochastic discount factor  $M$  is given by

$$M_t = V_x(t, X_t),$$

or equivalently by

$$M_t = U_c(t, c_t)$$

along the equilibrium path. In our present setting we have

$$V(t, x) = f(t, x, t, x),$$

so a conjecture would perhaps be that the stochastic discount factor for the time inconsistent case is given by at least one of the formulas

$$\begin{aligned} M_t &= V_x(t, X_t), \\ M_t &= f_x(t, X_t, t, X_t), \\ M_t &= U_c(t, X_t, t, c_t) \end{aligned}$$

along the equilibrium path. In order to check if any of these formulas are correct we only have to compute the corresponding differential  $dM_t$  and check whether it satisfies (56). It is then easily seen that none of the formulas for  $M$  are correct. The situation is in thus more complicated and we now go on to derive the correct representation of the stochastic discount factor.

### 11.7.1 A representation formula for $M$

We now go back to the time inconsistent case with utility of the form

$$E_{t,x} \left[ \int_t^T U(t, x, s, c_s) ds \right].$$

We will, below, present a representation for the the stochastic discount factor  $M$  in the production economy, but first we need to introduce some new notation.

**Definition 11.2** *Let  $X$  be a (possibly vector valued) semimartingale and let  $Y$  be an optional process. For a  $C^2$  function  $f(x, y)$  we introduce the “partial stochastic differential”  $\partial_x$  by the formula*

$$\partial_x f(X_t, Y_t) = df(X_t, y), \quad \text{evaluated at } y = Y_t. \quad (57)$$

The intuitive interpretation of this is that

$$\partial_x f(X_t, Y_t) = f(X_{t+dt}, Y_t) - f(X_t, Y_t), \quad (58)$$

and we have the following proposition, which generalizes the standard result for the time consistent theory.

**Theorem 11.1** *The stochastic discount factor  $M$  is determined by*

$$d(\ln M_t) = \partial_{t,x} \ln f_x(t, X_t, t, X_t), \quad (59)$$

where the partial differential  $\partial_{t,x}$  only operates on the variables  $(t, x)$  in  $f_x(t, x, s, y)$ . Alternatively we can write

$$M_t = U_c(t, X_t, t, \hat{c}_t) \cdot e^{Z_t}, \quad (60)$$

where  $Z$  is determined by

$$dZ_t = \partial_{t,x} \ln f_x(t, X_t, t, X_t) - d \ln f_x(t, X_t, t, X_t). \quad (61)$$

**Remark 11.1** *We remark here on the structural similarity of the stochastic discount factor to the result obtained in [12].*

**Remark 11.2** *For a more concrete interpretation of this result, see Section 11.7.2 below. Note again that the operator  $\partial_{tx}$  in (61) only acts on the first occurrence of  $t$  and  $X_t$  in  $f_x(t, X_t, t, X_t)$  whereas the operator  $d$  acts on the entire process  $t \mapsto f_x(t, X_t, t, X_t)$ .*

**Proof.** Formulas (60)-(61) follows from (59) and the first order condition  $U_c(t, X_t, t, \hat{c}_t) = f_x(t, X_t, t, X_t)$ . It thus remains to prove (59).

From (56) it follows that we need to show that

$$\partial_{t,x} \ln f_x(t, X_t, t, X_t) = - \left\{ r_t + \frac{1}{2} \varphi_t^2 \right\} dt + \varphi_t dW_t \quad (62)$$

where  $r$  and  $\varphi$  are given by (43)-(44). Applying Ito and the definition of  $\partial_{t,x}$  we obtain

$$\partial_{t,x} \ln f_x(t, X_t, t, X_t) = A(t, X_t)dt + B(t, X_t)dW_t,$$

where

$$A(t, x) = \frac{1}{f_x} \left\{ f_{xt} + (\alpha x - \hat{c}) f_{xx} + \frac{1}{2} \sigma^2 x^2 f_{xxx} - \frac{1}{2} \sigma^2 x^2 \frac{f_{xx}^2}{f_x} \right\}, \quad (63)$$

$$B(t, x) = \sigma x \frac{f_{xx}}{f_x}. \quad (64)$$

From (44) we see that indeed  $B(t, x) = \varphi(t, x)$  so, using (43), it remains to show that

$$A(t, x) = - \left\{ \alpha + \sigma^2 x \frac{f_{xx}}{f_x} + \frac{1}{2} \sigma^2 x^2 \frac{f_{xx}}{f_x} \right\}. \quad (65)$$

To show this we differentiate the equilibrium HJB equation (45), use the first order condition  $U_c = f_x$ , and obtain

$$\begin{aligned} U_x + f_{ty} + f_{tx} + (\alpha x - \hat{c}) f_{xx} + (\alpha x - \hat{c}) f_{xy} + \alpha f_x \\ + \sigma^2 x f_{xx} + \frac{1}{2} \sigma^2 x^2 f_{xxy} + \frac{1}{2} \sigma^2 x^2 f_{xxx} = 0, \end{aligned} \quad (66)$$

where  $f_{tx} = f_{tx}(t, x, t, x)$  and similarly for other derivatives,  $\hat{c} = \hat{c}(t, x)$  and  $U_x = U_x(t, x, t, \hat{c}(t, x))$ . From the extended HJB system we also recall the PDE for  $f^{sy}$  as

$$f_t^{sy}(t, x) + (\alpha x - \hat{c}) f_x^{sy}(t, x) + \frac{1}{2} \sigma^2 x^2 f_{xx}^{sy}(t, x) + U(s, y, t, \hat{c}) = 0$$

Differentiating this equation w.r.t. the variable  $y$  and evaluating at  $(t, x, t, x)$  and  $\hat{c}(t, x)$  we obtain

$$f_{ty} + (\alpha x - \hat{c}) f_{xy} + \frac{1}{2} \sigma^2 x^2 f_{xxy} + U_x = 0. \quad (67)$$

We can now plug this into (66) to obtain

$$f_{tx} + (\alpha x - \hat{c}) f_{xx} + \alpha f_x + \sigma^2 x f_{xx} + \frac{1}{2} \sigma^2 x^2 f_{xxx} = 0.$$

Plugging this into (63) we can write  $A$  as

$$A(t, x) = - \left\{ \alpha + \sigma^2 x \frac{f_{xx}}{f_x} + \frac{1}{2} \sigma^2 x^2 \frac{f_{xx}^2}{f_x} \right\}$$

which is exactly (65). ■

### 11.7.2 Interpreting the representation formula

The representation formula (59) does not *prima facie* seem to have a natural interpretation. We can of course write  $M$  as

$$M_t = e^{\int_0^t \partial_{t,x} \ln f_x(s, X_s, s, X_s) ds}$$

but this does not seem to give much insight. In order to get a deeper understanding we recall that for any (non dividend) asset price process  $p$  we have the valuation formula

$$p_s = E^P \left[ \frac{M_t}{M_s} p_t \middle| \mathcal{F}_t \right].$$

it is therefore natural to make the following definition.

**Definition 11.3** For any  $s < t$  we define the  $(s, t)$ -stochastic discount factor  $M_{st}$  by

$$M_{st} = \frac{M_t}{M_s}. \quad (68)$$

We thus have a natural multiplicative structure in the sense that for  $s < u < t$  we have

$$M_{st} = M_{su} \cdot M_{ut}.$$

The economic interpretation of  $M_{st}$  is thus that (via conditional expectation) it discounts the value of a stochastic claim at time  $t$  back to time  $s$ . It is now natural to look at the infinitesimal version of  $M_{st}$ . This object would intuitively be defined by the formula

$$M_{t,t+dt} = \frac{M_{t+dt}}{M_t} \quad (69)$$

and it would tell us how we discount on the infinitesimal scale from time  $t + dt$  back to time  $t$ .

In order to make (69) more precise we note that we can write it as

$$M_{t,t+dt} = e^{\ln M_{t+dt} - \ln M_t} = e^{d(\ln M_t)}$$

and this motivates the following formal definition.

**Definition 11.4** The log stochastic discount factor  $m_t$  is defined by

$$m_t = \ln M_t \quad (70)$$

We thus have

$$M_{st} = e^{m_t - m_s}$$

and the informal interpretation

$$M_{t,t+dt} = e^{dm_t}.$$

Theorem 11.1 shows that

$$dm_t = \partial_{t,x} \ln f_x(t, X_t, t, X_t),$$

so we have the interpretation

$$M_{t,t+dt} = e^{\partial_{t,x} \ln f_x(t, X_t, t, X_t)}$$

Using the interpretation (58) and doing some simple calculations we finally obtain the following (informal) result.

**Proposition 11.2** *With notation as above we have the following informal representation, corresponding to equation (59).*

$$\frac{M_{t+dt}}{M_t} = \frac{f_x(t+dt, X_{t+dt}, t, X_t)}{f_x(t, X_t, t, X_t)}. \quad (71)$$

This can also be written as

$$\frac{M_{t+dt}}{M_t} = \frac{U_c(t+dt, t+dt, X_{t+dt}, \hat{c}_{t+dt})}{U_c(t, t, X_t, \hat{c}_t)} \cdot \frac{f_x(t+dt, X_{t+dt}, t, X_t)}{f_x(t+dt, X_{t+dt}, t+dt, X_{t+dt})} \quad (72)$$

corresponding to equations (60)-(61).

Formula (71) has a natural economic interpretation which can be seen from a dimension argument. The valuation formula for a price process  $p$  will, on the infinitesimal scale, read as

$$p_t = E^P \left[ \frac{f_x(t+dt, X_{t+dt}, t, X_t)}{f_x(t, X_t, t, X_t)} p_{t+dt} \middle| \mathcal{F}_t \right]$$

If we denote the dimension of money by dollars we have the following relations, where  $dim$  denotes dimension

$$\begin{aligned} dim [f_x(t+dt, X_{t+dt}, t, X_t)] &= \text{marginal utility, at } t, \text{ of dollars at } t+dt. \\ dim [f_x(t, X_t, t, X_t)] &= \text{marginal utility, at } t, \text{ of dollars at } t. \end{aligned}$$

Since  $p_{t+dt}$  has dimension dollars at  $t+dt$ , we see that by multiplying with the factor  $f_x(t, X_t, t+dt, X_{t+dt})$  transforms this dollar amount into marginal utility at time  $t$ . Dividing by  $f_x(t, X_t, t, X_t)$  gives us dollars at  $t$ , which is the dimension of  $p_t$ .

## 11.8 Production economy with non-exponential discounting

A case of particular interest occurs when the utility function is of the form

$$U(t, x, s, c_s) = \beta(s-t)U(c_s)$$

so the utility functional has the form

$$E_{t,x} \left[ \int_t^T \beta(s-t)U(c_s)ds \right].$$

### 11.8.1 Generalities

In the case of non exponential discounting it is natural to consider the case with infinite horizon. We will thus assume that  $T = \infty$  so we have the functional

$$E_{t,x} \left[ \int_t^\infty \beta(\tau - t) U(c_\tau) d\tau \right]. \quad (73)$$

The function  $f(t, x, s, y)$  will now be of the form  $f(t, x, s)$  and, because of the time invariance, it is natural to look for time invariant equilibria where

$$\begin{aligned} \hat{\mathbf{u}}(t, x) &= \hat{\mathbf{u}}(x), \\ V(t, x) &= V(x), \\ f(t, x, s) &= g(t - s, x), \\ V(x) &= g(0, x). \end{aligned}$$

Observing that  $f_x(t, x, t) = g_x(0, x) = V_x(x)$  and similarly for second order derivatives, we may now restate proposition 11.1.

**Proposition 11.3** *With assumptions as above the following hold.*

- The equilibrium short rate is given by

$$r(x) = \alpha + \sigma^2 \frac{x V_{xx}(x)}{V_x(x)}. \quad (74)$$

- The equilibrium Girsanov kernel  $\varphi$  is given by

$$\varphi(x) = \sigma \frac{x V_{xx}(x)}{V_x(x)}. \quad (75)$$

- The extended equilibrium HJB system has the form

$$U(\hat{c}) + g_t(0, x) + (\alpha x - \hat{c}) g_x(0, x) + \frac{1}{2} x^2 \sigma^2 g_{xx}(0, x) = 0, \quad (76)$$

$$\mathbf{A}^{\hat{c}} g(t, x) + \beta(t) U(\hat{c}(x)) = 0, \quad (77)$$

- The function  $g$  has the representation

$$g(t, x) = E_{0,x} \left[ \int_0^\infty \beta(t+s) U(\hat{c}_s) ds \right] \quad (78)$$

- The equilibrium consumption  $\hat{c}$  is determined by the first order condition

$$U_c(\hat{c}) = g_x(0, x) \quad (79)$$

- The term  $\mathbf{A}^{\hat{c}}g(t, x)$  is given by

$$\mathbf{A}^{\hat{c}}g(t, x) = g_t(t, x) + x[\alpha - \hat{c}(x)]g_x(t, x) + \frac{1}{2}x^2\sigma^2g_{xx}(t, x) \quad (80)$$

- The equilibrium  $X$  dynamics are given by

$$dX_t = (\alpha X_t - \hat{c}_t) dt + X_t \sigma dW_t. \quad (81)$$

We see that the short rate  $r$  and the Girsanov kernel  $\varphi$  has exactly the same structural form as the standard case formulas (51)-(52). We now move to the stochastic discount factor and after some calculations we have the following version of Theorem 11.1.

**Proposition 11.4** *The stochastic discount factor  $M$  is determined by*

$$d \ln(M_t) = d \ln g_x(t, X_t) \quad (82)$$

where  $g_x$  is evaluated at  $(0, X_t)$ . Alternatively, we can write  $M$  as

$$M_t = U_c(\hat{c}_t) \cdot \exp \left\{ \int_0^t \frac{g_{xt}(0, X_s)}{g_x(0, X_s)} ds \right\} \quad (83)$$

We can also refer to Proposition 11.2 and conclude that the infinitesimal SDF is given by the formula

$$\frac{M_{t+dt}}{M_t} = \frac{g_x(dt, X_{t+dt})}{g_x(0, X_t)}, \quad (84)$$

or, alternatively, by the formula

$$\frac{M_{t+dt}}{M_t} = \frac{U_c(\hat{c}_{t+dt})}{U_c(\hat{c}_t)} \cdot \frac{g_x(dt, X_{t+dt})}{g_x(0, X_{t+dt})}. \quad (85)$$

### 11.8.2 Log utility

We now specialize to the case of log utility, so the utility functional has the form

$$E_{t,x} \left[ \int_t^\infty \beta(\tau - t) \ln(c_\tau) d\tau \right]. \quad (86)$$

Given some experience from the standard time consistent case, we now make the Ansatz

$$g(t, x) = a_t \ln(x) + b_t, \quad (87)$$

where  $a$  and  $b$  are deterministic functions of time. The natural boundary conditions are

$$\lim_{t \rightarrow \infty} a_t = 0, \quad \lim_{t \rightarrow \infty} b_t = 0. \quad (88)$$



With this Ansatz we have

$$g_t = \dot{a} \ln(x) + \dot{b}, \quad g_x = \frac{a}{x}, \quad g_{xx} = -\frac{a}{x^2}, \quad (89)$$

so from the first order condition (79) for  $c$  we obtain

$$\hat{c}(x) = \frac{x}{a_0}. \quad (90)$$

From Proposition 11.3 we obtain the short rate and the Girsanov kernel as

$$r = \alpha - \sigma^2, \quad \varphi = -\sigma. \quad (91)$$

The function  $a$  and  $b$  are determined by (77). We obtain

$$\dot{a} \ln(x) + \dot{b} + a\sigma^2 + (\alpha - \sigma^2)a - \frac{a}{a_0} - \frac{\sigma^2}{2}a + \beta \ln(x) - \beta \ln(a_0) = 0$$

We thus obtain the ODE

$$\dot{a}_t = -\beta(t), \quad (92)$$

and with the boundary condition  $\lim_{t \rightarrow \infty} a_t = 0$  this gives us

$$a_t = \int_t^\infty \beta(s) ds. \quad (93)$$

We also have an obvious ODE for  $b$ , but this is of little interest for us.

In order to determine the SDF we use Proposition 11.4 and compute

$$\frac{g_{xt}(0, x)}{g_x(0, x)} = \frac{\dot{a}_0}{a_0} = -\frac{1}{a_0}.$$

and we have the following result.

**Proposition 11.5** *For the case of log utility, the stochastic discount factor is given by*

$$M_t = \frac{1}{X_t} \cdot e^{-t/a_0}. \quad (94)$$

where

$$a_0 = \int_0^\infty \beta(s) ds.$$

### 11.8.3 Power utility

We now turn to the more complicate case of power utility so we have

$$U(c) = \frac{c^\gamma}{\gamma}$$

where  $\gamma < 1$ . We make the obvious Ansatz

$$g(t, x) = a_t \frac{x^\gamma}{\gamma} \quad (95)$$

We readily obtain

$$g_t = \dot{a} \frac{x^\gamma}{\gamma}, \quad g_x = ax^{\gamma-1}, \quad g_{xx} = a(\gamma-1)x^{\gamma-2}, \quad g_{xt} = \dot{a}x^{\gamma-1} \quad (96)$$

The first order condition for  $c$  is

$$c^{\gamma-1} = a_0x^{\gamma-1}$$

so the equilibrium consumption is given by

$$\hat{c}(x) = Dx \quad (97)$$

where

$$D = a_0^{-1/(1-\gamma)}$$

From Proposition 11.3 we obtain the short rate and the Girsanov kernel as

$$\begin{aligned} \varphi &= -\sigma(1-\gamma), \\ r &= \alpha - \sigma^2(1-\gamma). \end{aligned}$$

The function  $a$  is again determined by (77). We obtain

$$\dot{a} \frac{x^\gamma}{\gamma} + xa\sigma^2(1-\gamma)x^{\gamma-1} + xa(r-D)x^{\gamma-1} + a\frac{\sigma^2}{2}x^2(\gamma-1)x^{\gamma-2} + \beta \frac{D^\gamma x^\gamma}{\gamma} = 0$$

which gives us the linear ODE

$$\dot{a}_t + Aa_t + B\beta(t) = 0.$$

with

$$\begin{aligned} A &= \alpha - D - \frac{\sigma^2}{2}(1-\gamma), \\ B &= \frac{D^\gamma}{\gamma} \end{aligned}$$

The function  $a$  is thus given by

$$a_t = a_0e^{-At} - B \int_0^t e^{-A(t-s)}\beta(s)ds. \quad (98)$$

Using Proposition 11.4 we thus have the following result.

**Proposition 11.6** *For the case of power utility the stochastic discount factor is given by*

$$M_t = X_t^{\gamma-1} e^{\frac{a_0}{a_0}t}$$

From these examples with non exponential discounting we see that the risk free rate and Girsanov kernel only depend on the production opportunities in the economy. These objects are unaffected by the time inconsistency stemming from non-exponential discounting. Equilibrium consumption, however, is determined by the discounting function of the representative agent. In particular, we see that non exponential discounting has an effect on the marginal propensity to consume from wealth and thus affects the equilibrium level of wealth in the economy.

## 12 Conclusion and future research

In this paper we have presented a fairly general class of time inconsistent stochastic control problems. Using a game theoretic perspective we have derived a system of equations for the determination of the subgame perfect Nash equilibrium control, as well as for the corresponding equilibrium value function. The system is an extension of the standard dynamic programming equation for time consistent problems. We have studied a couple of concrete examples, and in particular we have studied the effect of time inconsistency in the framework of general equilibrium for a production economy.

Some obvious open research problems are the following.

- In Section 4.1 we informally derived the continuous time extended HJB system as a limit using a discretization argument. It would obviously be valuable to have a theorem which rigorously proves convergence of the discrete time theory to the continuous time limit. For the quadratic case, this is done in [6], but the general problem is completely open (and probably very hard).
- A related (hard) open problem is to prove existence and/or uniqueness for solutions of the extended HJB system.
- The present theory depends critically on the Markovian structure of the model. It would be interesting to see what can be one without this assumption.
- The equilibrium model in Section 11 can be extended to a multidimensional model with several underlying factors. This is the subject of a forthcoming paper.

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