Should Investors Learn about the Timing of Equity Risk?*

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Abstract

The term structure of equity risk has been documented to be downward-sloping. We capture this feature using return dynamics driven by both a transitory and a permanent component. We study the asset allocation and portfolio performance when transitory and permanent components cannot be observed, and therefore need to be estimated. Both in-sample and out-of-sample, strategies that account for the observed shape of the term structure of equity risk significantly outperform those which do not. Indeed, certainty equivalent returns are about 20% larger because properly modeling the timing of risk implies surges in portfolio returns.

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1 Introduction

Empirical research has recently provided evidence of a downward-sloping term structure of equity risk (van Binsbergen, Brandt, and Koijen, 2012). This result is groundbreaking because it challenges most models in the asset pricing and portfolio selection literature. Indeed, asset-pricing models typically consider a *permanent component* with stochastic drift and stochastic volatility in consumption growth (Bansal and Yaron, 2004), whereas portfolio selection models consider the same features for equity returns (Kim and Omberg, 1996; Wachter, 2002; Chacko and Viceira, 2005). These two features imply a markedly upward-sloping term structure of equity risk.

In this paper, we do not investigate the foundations of downward-sloping equity risk. Instead, we focus on its implications for investors. Namely, we provide a simple and flexible model of equity returns that allows the slope of the term structures of equity risk premia and return volatility to be either positive or negative. The key ingredient generating the downward-sloping effect is the *transitory component* of equity returns, which is ignored by most models.[1] This transitory component significantly increases equity risk in the short term and has little influence on it in the long term because it is mean-reverting. When the transitory component reverts sufficiently rapidly, its downward-sloping effect dominates the upward-sloping effect implied by both the stochastic drift and volatility in the permanent component, generating a downward-sloping term structure of equity risk.

We show that properly modeling the shape of the term structure of equity risk is absolutely crucial to investors because it significantly improves the performance of their investment strategies. Indeed, the *out-of-sample* mean and median certainty equivalent return of a strategy considering the transitory component is about 60% and 20% higher, respectively, than that of a strategy ignoring the transitory component. Accounting for the transitory component has no significant impact on the left tail of the distribution of portfolio returns, whereas it markedly lengthens and fattens its right tail providing investors with more attractive returns on their wealth. Importantly, we show that the performance advantages of learning about the timing of equity risk are larger in market downturns.

To investigate the implications of learning about the timing of equity risk, we consider a dynamic portfolio choice problem of an investor who has CRRA preferences and maximizes expected utility of terminal wealth. The investor can invest in two securities: one stock and one risk-free asset. Stock returns feature two standard ingredients, a *permanent component* with stochastic drift (Kim and Omberg, 1996) and stochastic volatility (Heston, 1993). In

[1] A few exceptions that consider a transitory component in either equity returns or dividend growth are Fama and French (1988) and, more recently, Menzly, Santos, and Veronesi (2004), Bansal, Kiku, and Yaron (2010), Greenwald, Lettau, and Ludvigson (2014), and Marfè (2015a, 2016).

addition, returns are driven by a *transitory component* that allows the term structure of equity risk to be potentially downward-sloping. Since the investor observes neither the permanent component nor the transitory component, she filters them out by combining Bayesian updating techniques with the observation of past returns. Learning dynamically about the relative importance of the two components allows the investor to account for changes in horizon-specific risk levels.

We fit the model to historical monthly S&P 500 returns using maximum-likelihood estimation. Estimated parameter values show that the transitory component is persistent, that the stochastic drift of the permanent component is a quickly reverting and highly volatile process, and that the model accurately replicates the observed downward-sloping term structure of S&P 500 return risk. Furthermore, we provide evidence that the transitory component strongly negatively covaries with the S&P 500 dividend yield (65% R^2), whereas the stochastic drift does only to a very limited extent (5% R^2). Since the dividend yield is a persistent forward-looking measure of equity compensation (van Binsbergen and Koijen, 2010), so is the transitory component.

Using the estimated parameter values, we investigate the properties of investor's hedging demands. Since returns are negatively correlated with volatility and positively correlated with the stochastic drift of the permanent component, both variables command a negative hedging term. The reason is that, in bad times (i.e. when returns are low), volatility is high and the expected return tends to be low because the stochastic drift is low. Since our investor is more risk averse than a log-utility investor, she hedges these two risks by reducing her investment in the stock. In contrast, the transitory component commands a positive hedging term because it is positively correlated with returns and it negatively impact expected returns. Indeed, the transitory component is small in bad times and therefore implies a high expected return, which our investor exploits by increasing her position in the stock.

As the investor's investment horizon increases, the size of the hedging terms tends to increase. The reason is that the model-implied term structure of equity risk features two upward-sloping components and one downward-sloping component. The upward slope is implied by the stochastic drift and stochastic volatility, whereas the downward slope is implied by the transitory component. Since the stochastic drift and volatility generate more risk in the long term than in the short term, the investor hedges these two risks more when her horizon is large. At the same time, the investor exploits the small amount of long-term risk implied by the transitory component and therefore increases her risky position when her horizon increases.

We compare performance of investment strategies that account for the transitory component of returns to that of those ignoring it. We show that modeling the transitory component of returns has a minor impact on the Sharpe ratio of portfolio returns but significantly increases their skewness, good relative to bad volatility, and good relative to bad kurtosis[2]. In addition, we provide evidence that modeling the transitory component is particularly beneficial to the investor who takes into account stochastic volatility. These results hold both in-sample and out-of-sample and are robust to changes in the investment horizon and risk aversion.

To provide a utility-based measure of the benefits of modeling the transitory component, we compare the mean certainty equivalent return (CER)[3] of strategies considering the transitory component to those ignoring it. Both in-sample and out-of-sample, accounting for the transitory component weakly impacts the CER when return volatility is assumed to be constant. However, when return volatility is assumed to be stochastic, including the transitory component increases the CER by about 10% and 20% in-sample and out-of-sample, respectively. Since time-varying return volatility is a well-known feature of the data (Engle, 1982; Bollerslev, 1986), our results show that ignoring the transitory component of returns is very costly for investors. Furthermore, we show that accounting for the transitory component increases the CER more when the coefficient of relative risk aversion is small and the investment horizon is large. The reason is that the investor's position in the risky asset is large when her risk aversion is small, but also when her horizon is long because there is less return volatility in the long run than in the short run.

Finally, we provide evidence and explore the implications of the fact that the observed term structure of equity risk is more downward-sloping in bad times than in good times, which is consistent with the findings by van Binsbergen, Hueskes, Koijen, and Vrugt (2013) and Aït-Sahalia, Karaman, and Mancini (2015). In our out-of-sample analysis, the investor re-estimates the parameters of the model every year and therefore captures this feature of the data when considering the transitory component in her model but not when ignoring it. As a result, the improvement in portfolio performance implied by modeling the transitory component is expected to be concentrated in bad times. We confirm this statement by showing that the unconditional benefits of considering the transitory component, in terms of both portfolio return moments and certainty equivalent returns, mostly come from the significantly better performance of such strategies in recessions.

This paper builds on the influential empirical studies of van Binsbergen et al. (2012) and

[2]Good and bad volatility (resp. good and bad kurtosis) is defined as the volatility (resp. kurtosis) conditional on returns being larger than their mean (Segal, Shaliastovich, and Yaron, 2015).

[3]We follow Johannes, Korteweg, and Polson (2014) and define the certainty equivalent return as the yield of a fictitious riskless asset that makes the investor indifferent between implementing her optimal risky strategy and investing her entire wealth in this riskless asset.

van Binsbergen et al. (2013), which show that the term structures of equity return volatility and risk premia are downward-sloping. Similarly, Dechow, Sloan, and Soliman (2004) provide evidence of a positive (short) duration premium in the cross-section and, more recently, Weber (2016) and Marfè (2015b) offer additional empirical evidence. Since the most successful asset-pricing models (e.g. Campbell and Cochrane, 1999; Bansal and Yaron, 2004; Gabaix, 2012; Wachter, 2013) imply markedly upward-sloping term structures of equity returns, the findings of van Binsbergen, Koijen, and co-authors have highlighted new asset-pricing puzzles that researchers have focused on since then. Among them, Belo, Collin-Dufresne, and Goldstein (2015), Croce, Lettau, and Ludvigson (2015), and Marfè (2016) provide foundations for the observed term structures of equity risk premia and volatility based on financial leverage, learning, and labour relations, respectively. Refer to van Binsbergen and Koijen (2015) for an excellent survey of this research stream.

The above-mentioned literature on the term structure of equity is silent about the implications of the timing of risk on asset allocation and portfolio performance. The goal of our paper is therefore to investigate the latter. In a related paper, Campbell and Viceira (2005) use a VAR model to study asset returns across different investment horizons. They show that the term structure of bond return volatility is less downward-sloping than that of equity return volatility, whereas the term structure of T-Bill return volatility is upward-sloping. Based on this evidence, Campbell and Viceira point out that if the term structure of risk is not flat the myopic mean-variance portfolio (Markowitz, 1952) is not optimal for long-horizon investors. We depart from the static mean-variance problem discussed in Campbell and Viceira (2005) along a number of important dimensions. In our paper, we propose a model of stock returns that allows for a flexible shape of the term structure of equity risk. We then study a full-fledged dynamic learning and portfolio choice problem of a CRRA investor. This setting allows us to discuss in- and out-of sample higher order moments and utility-based measures for a comprehensive examination of the effects of the timing of equity risk on portfolio performance.

Our paper is also related to the dynamic asset allocation literature with learning. Xia (2001) investigates the optimal portfolio allocation of an investor who faces uncertainty about return predictability. That is, the investor knows that some observable variable predicts future stock returns but is uncertain about the strength of the relation and therefore learns about it. Johannes et al. (2014) extend the setting of Xia (2001) to allow for parameter uncertainty and show that the out-of-sample performance of the associated strategy can be significantly better than that obtained with a model assuming constant mean and constant volatility of returns. We contribute to this literature by first showing that simple learning models without predictive variables generate significantly better out-of-sample portfolio performance than that of the constant mean and constant volatility model. More importantly, we provide evidence that properly learning about the shape of the term structure of equity risk significantly improves portfolio performance in terms of both portfolio return moments and certainty equivalent returns. These benefits turn out to be particularly large in bad times and under the assumption of stochastic return volatility.

The remainder of the paper is organized as follows. Section 2 provides empirical support for the assumptions of the model; Section 3 describes the model; Section 4 estimates the parameters of the model; Section 5 discusses the results; and Section 6 concludes.

2 Empirical Support

The aim of this section is to investigate equity risk across investment horizons. Time-variation in stock returns leads to horizon-specific risk levels. The shape of equity risk over different horizons is informative about the role of the transitory and permanent components of equity risk. We document movements in the slope of the observed term structure and their relation to the dividend yield.

We measure the timing of equity risk by computing the term structure of stock returns variance ratios (VRs hereafter), i.e. the ratio of the annualized returns variance at horizon τ relative to the annualized returns variance at the reference horizon τ_0:

$$\mathrm{VR}(\tau) = \frac{\mathrm{Var}(R_{t\to t+\tau})/\tau}{\mathrm{Var}(R_{t\to t+\tau_0})/\tau_0}.$$

VRs capture whether the variance of a given variable increases linearly with the observation interval. Therefore, downward-sloping VRs below unity imply that risk concentrates at short horizons. In contrast, upward-sloping VRs above unity imply that risk concentrates at long horizons. If stock returns variation is mostly due to shocks that have a permanent impact, we expect increasing VRs. If stock returns variation is mostly due to transitory risk, we expect decreasing VRs. If stock returns are either close to i.i.d. or the permanent and transitory components offset each other we expect approximatively flat VRs.

We consider monthly S&P 500 nominal log-returns from 01/1872 to 12/2015. Data are from Robert Shiller's website. We compute variance ratios from monthly data and use the 1-year variance as reference level.[4]

The left panel of Figure 1 depicts the term structures of VRs for horizons up to 10 years. We compute VRs for the full sample (1872-2015) as well as for the pre-war and post-war

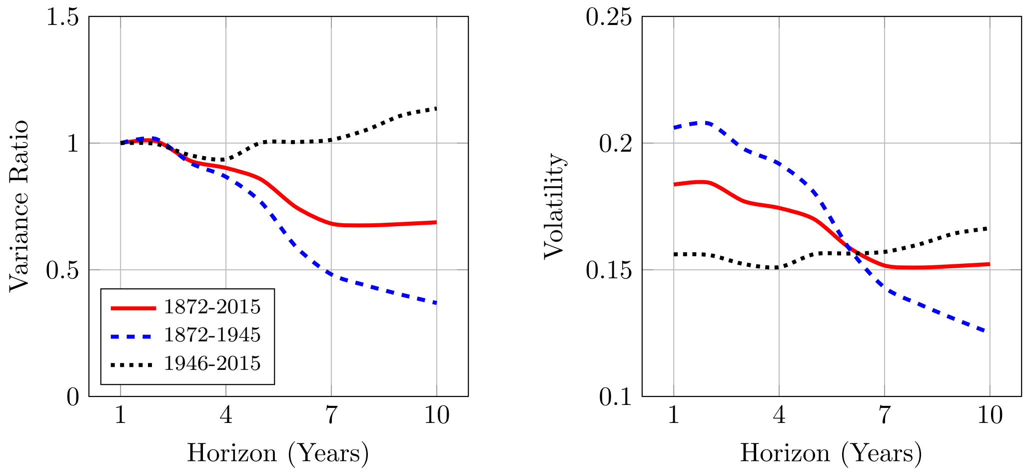
[4] Very similar results obtain computing the variance ratios from yearly returns aggregated from monthly data.

sub-samples (1871-1945 and 1946-2015). We easily observe that the slope of equity risk is either negative or about flat.

First, the full sample VRs are downward-sloping, the negative slope concentrates at the horizons between 3 and 7 years, and at longer horizons the VRs are about 0.70. Second, pre-war VRs are markedly downward-sloping, the negative slope starts at the 3-year horizon and persists, and the 10-year VR is about 0.37. Third, post-war VRs are about flat up to the 7-year horizon, are then slightly upward-sloping, and reach a level of about 1.13 at the 10-year horizon.

Both in the full sample and in the sub-samples, the 2-year VRs are close to unity. This comes from the fact that stock returns exhibit short-term time series momentum (Moskowitz, Ooi, and Pedersen, 2010), whereas long-term reversal implies decreasing VRs between the 2-year and the 4-year horizons. Long-horizon VRs are larger than zero indicating that prices follow an integrated process, but they are below or close to unity indicating that the transitory component of equity risk either offsets or more than offsets the permanent component.

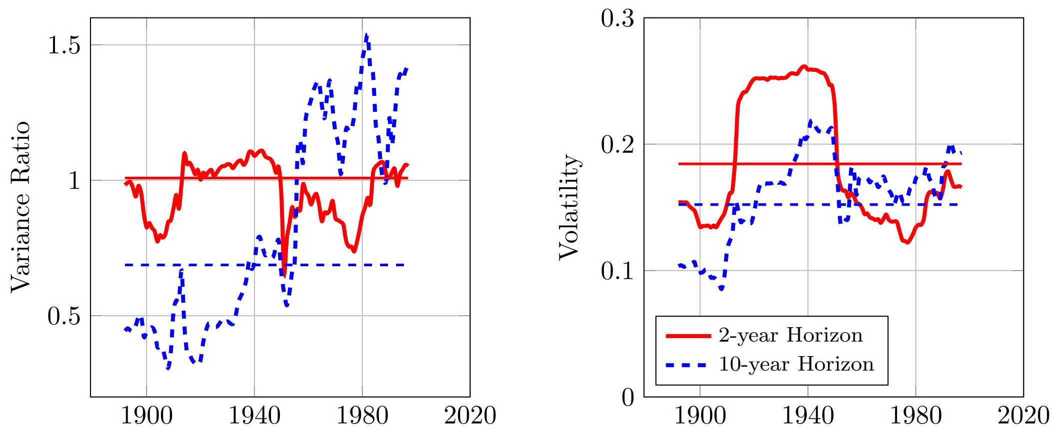
Figure 1: Term Structure of Variance Ratio and Volatility of Stock Returns. This figure displays the variance ratios and the volatilities up to 10 years of nominal S&P 500 yearly returns on several samples respectively in the left and right panels.



The right panel of Figure 1 reports the term structures of the annualized standard deviation of returns. In the full sample, return volatility decreases from about 18.5% at the 1-year horizon to about 15% at the 10-year horizon. In the pre-war sample, volatility decreases from 20.5% at the 1-year horizon to 12.5% at the 10-year horizon, whereas it increases from 15.5% at the 1-year horizon to 16.5% at the 10-year horizon in the post-war sample. Therefore, the increase in the slope of VRs from pre-war to post-war data is due to both a decrease in short-horizon risk and an increase in long-horizon risk.

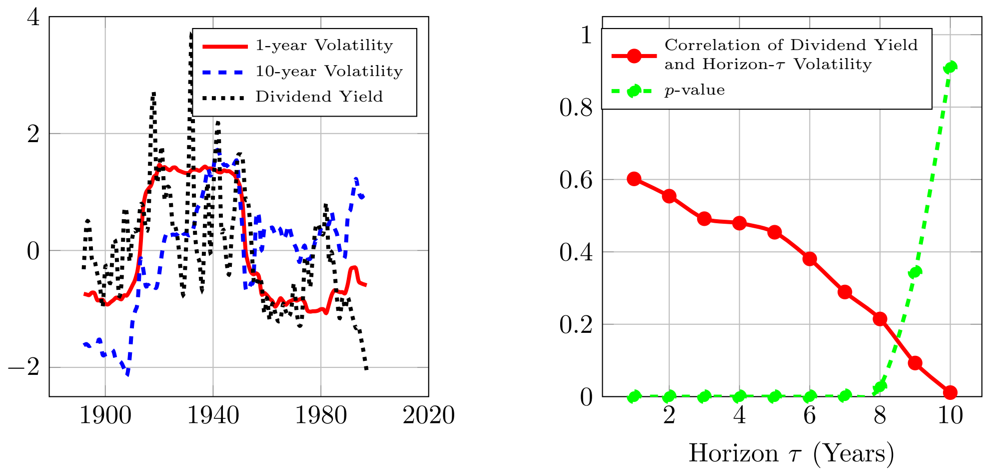
To further understand the timing of equity risk, we investigate time variation of stock returns VRs and annualized volatility. Using the full sample, we build time series at yearly frequency from a 40-year rolling window of monthly returns. This procedure gives 106 yearly observations of VRs and annualized volatility with horizons ranging from 1 to 10 years. The left and right panels of Figure 2 depict the time series of VRs and annualized volatility, respectively, at the 2-year and 10-year horizons. While the 2-year VRs fluctuate around the full sample average, the 10-year VRs show a secular positive trend and lie significantly below and above the full sample average in the pre-war and post-war samples, respectively. Such a trend is mostly due to the positive and similar pattern featured by the 10-year return volatility. In contrast, short-horizon volatility increases in the first two decades of the 20th century and then decreases in the 50's and 60's.

Figure 2: Variance Ratio and Volatility in the 20th Century.
This figure displays the variance ratios (left panel) and the volatilities (right panel) with 2-year and 10-year horizons of nominal S&P 500 yearly returns. Statistics are computed through a 40-year rolling window of monthly returns. Horizontal lines denote full sample statistics.



The left panel of Figure 3 displays the standardized time series of 1-year and 10-year return volatility together with the standardized dividend yield. The latter is a forward-looking measure of equity compensation and, hence, it is interesting to investigate how the dividend yield relates to short-horizon and long-horizon equity risk. While the positive trend in long-horizon volatility seems quite unrelated with time variation in the dividend yield, we observe that short-horizon volatility captures both the hump and the decline of the dividend yield in pre-war and post-war samples, respectively. The right panel of Figure 3 shows the correlations (and p-values) between the dividend yield and return volatility for horizons ranging from 1 to 10 years. While the correlations with short-horizon volatility are large and significant, those with long-horizon volatility decline toward zero and become insignificant.

Figure 3: Volatility and Dividend Yield.
The left panel displays the standardized dividend yield, the 1-year returns volatility and the 10-year returns volatility of S&P 500. The dividend yield is computed averaging monthly observations in each years and volatilities are computed through a 40-year rolling window of monthly returns. The right panel displays the correlation (and p-values) of the dividend yield with the S&P 500 return volatility computed at several horizons.



An implication of the previous result concerns the cyclicality of the slope of equity risk. While the negative slope appears as an unconditional property of the data, we can understand how the slope behaves dynamically by observing the contemporaneous and intertemporal relation between the long-horizon VRs and the dividend yield. The former represent a quantitative measure of the slope of equity risk and the latter represents a (counter-cyclical) measure of economic conditions. Since short-term return volatility is highly positively correlated with the dividend yield, we expect VRs to be negatively correlated with it.

Confirming our expectation, Table 1 shows that the contemporaneous correlation between the dividend yield and the 5-, 7-, and 10-year VRs are negative and large, ranging between -20% and -55%. This suggests that the slope of the term structure of equity risk is pro-cyclical. That is, consistently with the empirical findings of van Binsbergen et al. (2013) and Aït-Sahalia et al. (2015), short-term equity risk increases relative to long-term equity risk when economic conditions deteriorate. Table 1 also provides information about the intertemporal relation between the slope of the term structure of equity risk and the dividend yield. Long-horizon VRs are highly negatively correlated with future levels of the dividend yield. This implies that the slope of the term structure of equity risk forecasts the cyclical fluctuations in economic conditions and in expected market compensation.

The analysis of this section documents that the timing of equity risk is markedly downward-

Table 1: Variance Ratio and Dividend Yield.
This table reports the correlation between the variance ratios with 5, 7, and 10 years horizon of nominal S&P 500 yearly returns and both the contemporaneous and the future average dividend yield (dy). The variance ratios are computed through a 40-year rolling window of monthly returns and using a reference horizon τ_0 of one year. p-values are reported in parenthesis.

	dy_t	$\frac{1}{n}\sum_{i=1}^{n} dy_{t+i}$			
		1	3	5	7
VR(5)	-0.203	-0.228	-0.283	-0.319	-0.340
p-value	(0.037)	(0.019)	(0.003)	(0.001)	(0.000)
VR(7)	-0.502	-0.526	-0.596	-0.633	-0.656
p-value	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
VR(10)	-0.543	-0.559	-0.626	-0.664	-0.690
p-value	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

sloping, confirming former results in Campbell and Viceira (2005).[5] That is, investing in equity over a 1-year horizon is significantly more risky than investing over a 5-year or a 10-year horizon. This pattern of the timing of equity risk is robust, but its magnitude has decreased over the 20th century. The analysis of the relation between the dividend yield and the return volatility computed over different horizons suggests that short-run equity risk is much more closely linked to the dividend yield—a proxy of the equity risk premium—than long-run equity risk. Importantly, the slope of equity risk is pro-cyclical (i.e. short-run risk raises in bad times) and forecasts the dividend yield. Thus, disentangling permanent and transitory components of equity risk is key to understand risk-adjusted returns.

The short-run nature of equity risk documented above is in line with the recent findings about dividend strips (van Binsbergen et al., 2012, 2013) and payouts to shareholders (Belo et al., 2015; Marfè, 2016). Moreover, Koijen, Lustig, and van Nieuwerburgh (2014) also document the central role of a transitory (i.e. business cycle) component, as a priced state variable in the cross-section of financial returns. Overall this section suggests that accounting for a large and transitory component of stock returns is important to correctly understand, model, and estimate equity risk.

[5] Downward-sloping equity risk is a robust characteristics of U.S. stock market: a broader index than S& P 500 such as the value weighted CRSP index (including NYSE, Nasdaq and Amex stocks) features VR's declining from one to 48% and volatilities declining from 20% to 13.8% between the 1-year and 10-year horizons.

3 The Model

In this section, we first describe the economic setting in which an investor models stock returns using permanent and transitory components that are unobservable. Then, we solve the investor's learning problem and characterize her optimal portfolio allocation.

3.1 Investment Opportunities

We consider an investor who can invest in one risky asset—*a stock*—with price S_t and one riskless asset with constant risk-free interest rate r_f. The stock price, S_t, is modeled as the product of two terms:

$$S_t \equiv Y_t Z_t = S_0 e^{\int_0^t \left(\mu + x_u - \frac{1}{2}\sigma_u^2\right)du + \sigma_u dB_u^y} \times e^{z_t}, \tag{1}$$

where the dynamics of Y_t and Z_t satisfy

$$d\log Y_t = dy_t = \left(\mu + x_t - \sigma_t^2/2\right)dt + \sigma_t dB_t^y,$$
$$dx_t = -\kappa_x x_t dt + \sigma_x dB_t^x, \tag{2}$$
$$d\log Z_t = dz_t = -\kappa_z z_t dt + \sigma_z dB_t^z. \tag{3}$$

The Brownian motions B^y, B^x, and B^z are independent and the full filtration generated by observing all three Brownian shocks is denoted by $\mathbf{F}$.

The aim of considering the stock price as a product of the two components is to introduce some flexibility in modelling the timing of equity risk (Marfè, 2015a, 2016). The first term, Y, is an integrated process which depends on the time integral of x. That is, shocks in x accumulate and therefore permanently affect future prices. For this reason, we call x the *stochastic drift* of the stock return *permanent component*. If $Z_t \equiv 1$ we have the standard stock price dynamics with mean-reverting returns (Kim and Omberg, 1996; Wachter, 2002), which, as we will show, give rise to an upward-sloping term structure of equity risk because of the accumulation of shocks in x. The second term, Z_t, depends on the current value of z_t only. Since the process z is mean-reverting, shocks in z dissipate as time passes and therefore have a transitory impact on prices. For this reason, we call z the *transitory component* of stock returns.[6] By definition, the transitory component generates risk in the short term that dissipates in the longer term. That is, the transitory component helps to reconcile the timing of risk in the model with that observed empirically.

By an application of Ito's Lemma, the stock price dynamics satisfy

$$\frac{dS_t}{S_t} = (m + x_t - \kappa_z z_t)dt + \sqrt{v_t}dB_t, \tag{4}$$

where $m \equiv \mu + \sigma_z^2/2$ is the long run expected rate of return, $v_t \equiv \sigma_t^2 + \sigma_z^2$ is the conditional variance of the risky asset return (assumed to be either constant, $v_t \equiv \bar{v}$, or allowed to vary stochastically in time). Finally, $dB_t \equiv \left(\sigma_t dB_t^y + \sigma_z dB_t^z\right)/\sqrt{v_t}$ is an increment of a standard Brownian motion.

3.2 Investor's Learning Problem

The expected rate of return on the stock varies over time due to shocks that come from two sources: the permanent component x_t defined in (2) and the transitory component z_t defined in (3). The investor only has access to information generated by the observation of the realized returns defined in (4), and does not have access to the full information contained in the filtration $\mathbf{F}$. Therefore, all her actions must be adapted to her observation filtration $\mathbf{F}^S = \left\{\mathcal{F}_t^S\right\}_{t\geq 0}$, the flow of information generated by the stock price process S. In other words, the investor needs to filter out through Bayesian updating the *unobservable* factors x_t and z_t by observing the history of the stock price process only. Note that since the stock return variance v_t can be inferred from the quadratic variation of returns, it is also observable. We assume, however, that it does not provide more information on x_t and z_t than the price S_t does. This simplifying assumption implies that the dynamics of v_t do not enter the investor's learning problem.

Proposition 1 below provides the dynamics of the state variables projected onto the investor's observation filtration.

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[6] Note that the transitory component here is a different object from the transitory component in the sense of Alvarez and Jermann (2005) and Hansen and Scheinkman (2009). They show that any integrated process can be decomposed in a martingale and a transitory component. For instance, a geometric Brownian motion with mean-reverting drift can be written as the product of a martingale and a transitory process, which is driven by the mean-reverting drift and is negatively correlated with the martingale. However, since the drift is embedded in an integrated process, its variation accumulates over time and leads to upward-sloping risk. Instead, in this paper we refer to a transitory component which is not embedded in an integrated process, such that its variation does not accumulate over time and therefore leads to downward-sloping risk. Namely, we model a transitory component which is orthogonal to (instead of negatively correlated with) the integrated component of the cumulative return (and its martingale).

Proposition 1. *With respect to the investor's observation filtration, the dynamics of the stock price S_t, the filtered transitory component $\widehat{z}_t$, the filtered permanent component $\widehat{x}_t$, satisfy*

$$\frac{dS_t}{S_t} = (m + \widehat{x}_t - \kappa_z \widehat{z}_t)\, dt + \sqrt{v_t} d\widehat{B}_t, \tag{5}$$

$$d\widehat{x}_t = -\kappa_x \widehat{x}_t dt + \frac{1}{\sqrt{v_t}} (\gamma_{x,t} - \kappa_z \gamma_{zx,t})\, d\widehat{B}_t, \tag{6}$$

$$d\widehat{z}_t = -\kappa_z \widehat{z}_t dt + \frac{1}{\sqrt{v_t}} \left(\gamma_{zx,t} - \kappa_z \gamma_{z,t} + \sigma_z^2\right) d\widehat{B}_t, \tag{7}$$

where $\widehat{x}_t \equiv E_t\left(x_t | \mathcal{F}_t^S\right)$, $\widehat{z}_t \equiv E_t\left(z_t | \mathcal{F}_t^S\right)$, and $\widehat{B}_t$ is a $\mathcal{F}_t^S$-Brownian motion process. The posterior variance-covariance matrix Γ_t is defined as follows:

$$\Gamma(t) \equiv \begin{pmatrix} \gamma_{z,t} & \gamma_{zx,t} \\ \gamma_{zx,t} & \gamma_{x,t} \end{pmatrix} = \begin{pmatrix} \mathrm{Var}_t\left(z_t | \mathcal{F}_t^S\right) & \mathrm{Cov}_t\left(z_t, x_t | \mathcal{F}_t^S\right) \\ \mathrm{Cov}_t\left(z_t, x_t | \mathcal{F}_t^S\right) & \mathrm{Var}_t\left(x_t | \mathcal{F}_t^S\right) \end{pmatrix} \tag{8}$$

and the posterior variance-covariances $\gamma_{z,t}$, $\gamma_{x,t}$, $\gamma_{zx,t}$ are given by

$$\frac{d\gamma_{z,t}}{dt} = \sigma_z^2 - 2\kappa_z \gamma_{z,t} - \frac{1}{v_t} \left(\gamma_{zx,t} - \gamma_{z,t}\kappa_z + \sigma_z^2\right)^2, \tag{9}$$

$$\frac{d\gamma_{x,t}}{dt} = \sigma_x^2 - 2\kappa_x \gamma_{x,t} - \frac{1}{v_t} \left(\gamma_{x,t} - \gamma_{zx,t}\kappa_z\right)^2, \tag{10}$$

$$\frac{d\gamma_{zx,t}}{dt} = -\left(\kappa_x + \kappa_z\right) \gamma_{zx,t} - \frac{1}{v_t} \left(\gamma_{x,t} - \gamma_{zx,t}\kappa_z\right) \left(\gamma_{zx,t} - \gamma_{z,t}\kappa_z + \sigma_z^2\right). \tag{11}$$

Proof. See Liptser and Shiryaev (2001).

Equation (5) gives the return dynamics projected on the observable filtration. Equations (6)–(7) describe updating of the investor's expectation of the latent state variables x_t and z_t. We will refer to $\widehat{x}_t$ and $\widehat{z}_t$ as the filter estimates of the permanent component stochastic drift and the transitory component, respectively. Equations (9)–(10)–(11) provide the dynamics of posterior variance-covariance matrix (8) and hence capture the evolution of uncertainty associated with the estimation of the unobserved factors. Let us compare the true, but unobserved, dynamics of the state variables in (2)–(3) with the dynamics of filter estimates in (6)–(7), which represent investor's best guess of the latent states. We see that in the true model x_t and z_t are independent, while the investor's estimates are perfectly correlated. This is due to the fact that investor does not observe shocks to the latent variables directly, but only observes the price changes. This endogenous perfect correlation results in the instantaneous volatility of filter estimates being smaller than the true volatility of x_t and z_t.

3.3 Investor's Optimization Problem

The objective of the investor is to choose the fraction of her wealth invested in the stock π_t—*the portfolio weight*—that maximizes her expected utility of terminal wealth

$$\sup_{(\pi_s)_{s\in[t,T]}} E\left[e^{-\delta T}\frac{{W_T}^{1-\gamma}}{1-\gamma}\middle|\mathcal{F}_t^S\right], \tag{12}$$

where δ is the subjective discount rate, γ is the investor's relative risk aversion, and W is the investor's wealth that needs to satisfy the budget constraint

$$dW_t = r_f W_t dt + \pi_t W_t \left(m + \widehat{x}_t - \kappa_z \widehat{z}_t - r_f\right) dt + \pi_t W_t \sqrt{v_t} d\widehat{B}_t. \tag{13}$$

When the stock return variance is assumed to be constant, the investor's optimization problem given by Equations (12)–(13) has two state variables, the filter estimates of the two factors driving the stock price, $\widehat{x}_t$ and $\widehat{z}_t$. Accordingly, the value function in the constant volatility model ($v_t \equiv \bar{v}$) depends on investor's wealth and the two filter estimates. The uncertainty of the filter estimates does not become a state variable, because the posterior variance-covariance matrix of investor's updating rule, Γ_t, reaches a steady state. Its steady state value is obtained by setting the right-hand side of Equations (9)–(10)–(11) to zero in order to obtain the point at which the investor is no longer able to improve upon the accuracy of the estimates.

Proposition 2 below discusses the form of the value function and the optimal portfolio weight for the constant volatility case.

Proposition 2. *In the model with constant return volatility ($v_t \equiv \overline{v}$), the following hold.*

1. *The investor's optimal value function J takes the form*

$$J(t, W_t, \theta_t) = e^{-\delta t}\frac{W_t^{1-\gamma}}{1-\gamma} F(T-t, \theta_t)^{\gamma}, \tag{14}$$

where $\theta_t = [\widehat{x}_t, \widehat{z}_t]^\top$ and the function $F : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is given by

$$F(\tau, \theta) \equiv \exp\left(\frac{1}{2}\theta^\top A(\tau)\theta + \theta^\top B(\tau) + C(\tau)\right),$$

with A, B, C solving the system of matrix ODEs provided in Appendix A.1.

2. The optimal trading strategy of the investor is

$$\pi_t^\star = \frac{m + \widehat{x}_t - \kappa_z \widehat{z}_t - r_f}{\gamma \overline{v}} + \pi_t^{\text{hdg}} \tag{15}$$

with the hedging demand given by

$$\pi_t^{\text{hdg}} = \frac{\overline{\gamma}_x - \kappa_z \overline{\gamma}_{zx}}{\overline{v}} \frac{F_x}{F} + \frac{\overline{\gamma}_{zx} - \kappa_z \overline{\gamma}_z + \sigma_z^2}{\overline{v}} \frac{F_z}{F}, \tag{16}$$

where F and its derivatives are evaluated at $(T - t, \widehat{x}_t, \widehat{z}_t)$, and $\overline{\gamma}_x, \overline{\gamma}_z, \overline{\gamma}_{zx}$ are elements of the steady-state posterior variance covariance matrix $\overline{\Gamma}$ provided in Appendix A.1.

Proof. See Appendix A.1.

The optimal value function in (14) is separable in wealth with the function of state variables, F, having a familiar exponential quadratic form.

The optimal portfolio in (15) has two components: the myopic demand and the hedging demand. As $t \to T$, the investor's value function given in (14) approaches the terminal utility of wealth which is independent of the state vector $[\widehat{x}_t,\ \widehat{z}_t]^\top$. Therefore, the partial derivatives of the value function with respect to these state variables and consequently the hedging part (16) of the optimal portfolio will approach zero as horizon shrinks. This means that investors with short horizon will choose their stock allocation according to the myopic first term in (15). The hedging demand provides a long-horizon investor with an intertemporal hedge against changes in the two state variables, namely the filtered stochastic drift and transitory component of stock returns.

We now relax the assumption of constant volatility. In particular, we assume that stock return variance $v_t \in \mathcal{F}_t^S$ follows a square-root process (Heston, 1993):

$$dv_t = \kappa_v(\bar{v} - v_t)dt + \sigma_v \sqrt{v_t} dB_t^v, \tag{17}$$

where κ_v, $\bar{v}$, and σ_v are known constants, and where the shocks to variance are perfectly correlated with shocks to prices on the observable filtration, $d\langle B^v, \widehat{B}\rangle_t = dt$. Since σ_v can be either positive or negative, its sign determines the sign of the correlation between stock returns and variance. As we will see in Section 4, in line with results in Christoffersen, Jacobs, and Mimouni (2010), we estimate the correlation between returns and variance to be negative. It is important to note that the perfect correlation between the stock return, dS_t/S_t, and its variance, dv_t, on the observed filtration implies the same perfect correlation on the full filtration. That is, the return variance does indeed not, as assumed initially, provide more information on x_t and z_t than dS_t/S_t, and was therefore rightfully omitted from the learning problem (see Section 3.2).

In the model with stochastic volatility we no longer have the steady-state value for posterior variance-covariance matrix of filter estimates, which means that the filtering risk, as captured by elements of Γ_t in (8), enters as a state variable in investor's optimization problem. Proposition 3 below discusses the form of the value function and the optimal portfolio weight for the stochastic volatility case.

Proposition 3. *Assuming that the stock return variance is given by* (17)*, the following hold.*

1. *The investor's optimal value function J can be approximated by* [7]

$$J(t, W_t, \theta_t) \approx e^{-\delta t} \frac{W_t^{1-\gamma}}{1-\gamma} F(T-t, \theta_t)^{\gamma}, \tag{18}$$

where $\theta_t = [\widehat{x}_t, \widehat{z}_t, v_t, \gamma_{x,t}, \gamma_{z,t}, \gamma_{zx,t}]^{\top}$ and the function $F : \mathbb{R}^+ \times \mathbb{R}^6 \to \mathbb{R}$ is given by

$$F(\tau, \theta) \equiv \exp\left(\frac{1}{2}\theta^{\top} A(\tau)\theta + \theta^{\top} B(\tau) + C(\tau)\right),$$

with A, B, C solving the system of matrix ODEs defined in Appendix A.2.

2. *The optimal portfolio weight implied by the function J in* (18) *satisfies*

$$\pi_t^{\star} = \frac{m + \widehat{x}_t - \kappa_z \widehat{z}_t - r_f}{\gamma v_t} + \pi_t^{\text{hdg}}$$

with the hedging demand given by

$$\pi_t^{\text{hdg}} = -\sigma_v \frac{F_v}{F} + \frac{\gamma_{x,t} - \kappa_z \gamma_{zx,t}}{v_t} \frac{F_x}{F} + \frac{\gamma_{zx,t} - \kappa_z \gamma_{z,t} + \sigma_z^2}{v_t} \frac{F_z}{F},$$

where F and its derivatives are to be evaluated at $(T-t, \widehat{x}_t, \widehat{z}_t, v_t, \gamma_{x,t}, \gamma_{z,t}, \gamma_{zx,t})$.

Proof. See Appendix A.2.

Stochastic volatility induces an additional term to the hedging term of the optimal portfolio when compared to the constant volatility results in Proposition 2.

[7] The approximation method is borrowed from Andrei and Hasler (2015) who, in the spirit of Campbell and Shiller (1988) and Benzoni, Collin-Dufresne, and Goldstein (2011) approximations, linearize the partial differential equation satisfied by the function F. Evidence of the accuracy of the approximation is provided in Andrei, Carlin, and Hasler (2015).

4 Model Estimation

This section describes the Maximum-Likelihood estimation of the model parameters. We discuss different model sub-cases. Since all models are constrained versions of the general model with stochastic drift, stochastic variance, and a transitory component, we describe the estimation procedure for the general model only.

Based on the dynamics provided in (5), (6), (7), (9), (10), (11), and (17), the monthly log-return r_t, the transitory component $\widehat{z}_t$, the permanent component $\widehat{x}_t$, the variance v_t, and the posterior variance-covariances γ_z, γ_x, and γ_{zx} are discretized as follows:

$$
\begin{aligned}
r_{t+\Delta} &= (m - v_t/2 + \widehat{x}_t - \kappa_z \widehat{z}_t)\,\Delta + \sqrt{v_t \Delta}\epsilon_{t+\Delta},\\
\widehat{z}_{t+\Delta} &= e^{-\kappa_z \Delta}\widehat{z}_t + \left(\gamma_{zx} - \gamma_z \kappa_z + \sigma_z^2\right)\sqrt{\frac{1-e^{-2\kappa_z\Delta}}{2\kappa_z v_t}}\epsilon_{t+\Delta},\\
\widehat{x}_{t+\Delta} &= e^{-\kappa_x \Delta}\widehat{x}_t + \left(\gamma_x - \gamma_{zx}\kappa_z\right)\sqrt{\frac{1-e^{-2\kappa_x\Delta}}{2\kappa_x v_t}}\epsilon_{t+\Delta},\\
v_{t+\Delta} &= e^{-\kappa_v \Delta} v_t + \overline{v}\left(1 - e^{-\kappa_v\Delta}\right) + \operatorname{sign}(\sigma_v)\sqrt{l_t}\epsilon_{t+\Delta},\\
\gamma_{z,t+\Delta} &= \gamma_{z,t} + \left[-\frac{\left(\gamma_{zx,t} - \gamma_{z,t}\kappa_z + \sigma_z^2\right)^2}{v_t} - 2\gamma_{z,t}\kappa_z + \sigma_z^2\right]\Delta,\\
\gamma_{x,t+\Delta} &= \gamma_{x,t} + \left[-\frac{\left(\gamma_{x,t} - \gamma_{zx,t}\kappa_z\right)^2}{v_t} - 2\gamma_{x,t}\kappa_x + \sigma_x^2\right]\Delta,\\
\gamma_{zx,t+\Delta} &= \gamma_{zx,t} + \left[-\frac{\left(\gamma_{x,t} - \gamma_{zx,t}\kappa_z\right)\left(\gamma_{zx,t} - \gamma_{z,t}\kappa_z + \sigma_z^2\right)}{v_t} - \left(\kappa_x + \kappa_z\right)\gamma_{zx,t}\right]\Delta,
\end{aligned}
$$

where $\Delta = 1/12 = 1$ month, $\epsilon_t \sim \mathrm{N}(0,1)$ follows a Normal distribution with mean 0 and variance 1, and $l_t \equiv \frac{\sigma_v^2}{\kappa_v} v_t (e^{-\kappa_v\Delta} - e^{-2\kappa_v\Delta}) + \frac{\sigma_v^2}{2\kappa_v}\overline{v}(1 - e^{-\kappa_v\Delta})^2$. Conditional on knowing the parameters and the initial values $\widehat{z}_0$, $\widehat{x}_0$, v_0, $\gamma_{z,0}$, $\gamma_{x,0}$, and $\gamma_{zx,0}$, observing monthly returns r_t allows us to sequentially back out ϵ_t and therefore $\widehat{z}_t$, $\widehat{x}_t$, v_t, $\gamma_{z,t}$, $\gamma_{x,t}$, and $\gamma_{zx,t}$.

The 8-dimensional vector of parameters is $\Theta \equiv (m, \sigma_z, \kappa_z, \sigma_x, \kappa_x, \overline{v}, \sigma_v, \kappa_v)$ and the log-likelihood function $\mathcal{L}(\Theta; r_\Delta, r_{2\Delta}, \ldots, r_{N\Delta})$ satisfies

$$
\begin{aligned}
\mathcal{L}\left(\Theta; r_\Delta, r_{2\Delta}, \ldots, r_{N\Delta}\right) = \sum_{i=1}^{N} &\log\left(\frac{1}{\sqrt{2\pi v_t \Delta}}\right)\\
&- \frac{1}{2}\frac{\left(r_{i\Delta} - \left(m - \frac{1}{2}v_t + \widehat{x}_{(i-1)\Delta} - \kappa_z \widehat{z}_{(i-1)\Delta}\right)\Delta\right)^2}{v_t \Delta},
\end{aligned} \tag{19}
$$

where N is the number of observations. The vector of parameters Θ is chosen to maximize

Table 2: Parameter Values (Estimated over Entire Sample).
This table reports the estimated parameter values for the different models. P, PT, CV, SV, and CM stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant volatility, stochastic volatility, and constant mean, respectively. The Akaike information criterion is defined as follows: $AIC = 2k - 2\mathcal{L}$, where k is the number of parameters of the model and $\mathcal{L}$ is the corresponding log-likelihood function. Standard errors are reported in brackets and statistical significance at the 10%, 5%, and 1% levels is labeled with *, **, and ***, respectively. Significance for κ_x, κ_z, κ_v, and $\overline{v}$ are based on one-sided tests because these parameters are positive by definition. Significance for the other parameters are based on two-sided tests. Parameters are estimated using monthly S&P 500 returns from 02/1871 to 02/2016. The last three rows report our choice of preference parameters and the risk-free rate.

Parameter	**Symbol**	**CM-CV**	**P-CV**	**PT-CV**	**P-SV**	**PT-SV**
Long-term mean of	m	0.095***	0.094***	0.095***	0.091***	0.084***
expected return		(0.012)	(0.015)	(0.009)	(0.017)	(0.008)
Volatility of	σ_z			0.148***		0.171***
transitory component				(0.018)		(0.019)
Mean-reversion speed of	κ_z			0.149*		0.075***
transitory component				(0.112)		(0.022)
Volatility of	σ_x		3.168***	3.061***	1.016***	1.046***
the permanent component			(0.523)	(0.317)	(0.041)	(0.024)
Mean-reversion speed of	κ_x		30.066***	28.237***	7.214***	7.419***
the permanent component			(7.443)	(5.033)	(0.571)	(0.304)
Long-term mean of	$\overline{v}$	0.020***	0.018***	0.018***	0.017***	0.017***
return variance		(2.6×10^{-4})	(2.7×10^{-4})	(3.1×10^{-4})	(6.4×10^{-4})	(5.2×10^{-4})
Volatility of	σ_v				−0.042***	−0.042***
return variance					(0.002)	(0.003)
Mean-reversion speed of	κ_v				0.795***	0.895***
return variance					(0.076)	(0.094)
Akaike criterion	AIC	−6207.6	−6355.6	−6353.9	−6482.0	−6492.0
Relative risk aversion	γ	5	5	5	5	5
Subjective discount rate	δ	0.01	0.01	0.01	0.01	0.01
Risk-free rate	r^f	0.05	0.05	0.05	0.05	0.05

the log-likelihood function provided in Equation (19).

The constrained models are obtained by setting either constant mean ($\sigma_x = \kappa_x = \sigma_z = \kappa_z = 0$), constant variance ($v_t = \overline{v}$), or a permanent component only ($\sigma_z = \kappa_z = 0$). Each model is fitted to monthly S&P 500 nominal returns from 02/1871 to 02/2016. Data are from Robert Shiller's website.

Table 2 provides the estimated parameter values and their standard errors. Irrespective of whether return variance is assumed to be constant or stochastic, the transitory component $\widehat{z}_t$ features a relatively high volatility of about 15% and strong persistence (see bottom panel of Figure 4). Indeed, its half-life is about 4.5 and 9 years (i.e. business-cycle frequencies) when return variance is assumed to be constant and stochastic, respectively. The significance of the mean-reversion speed of the transitory component increases drastically when stochastic return variance is considered. As confirmed by the Akaike information criteria, this suggests that adding a transitory component increases the goodness of fit more when return variance is stochastic than when it is constant. In contrast with the transitory component, the stochastic drift $\widehat{x}_t$ mean-reverts at a high speed (see top panel of Figure 4). Its half-life is about 3 months and 8 months when return variance is constant and stochastic, respectively. It is worth noting that the volatility of return variance parameter σ_v is negative, implying a negative correlation between returns and return variance, and therefore reflecting the leverage effect (Christie, 1982; Christoffersen et al., 2010).

Focusing on the models' Akaike information criteria shows that considering a model with stochastic expected returns increases importantly the goodness of fit. Considering stochastic return variance increases the goodness of fit further and magnifies the benefits of considering a transitory component. When return variance is constant, however, considering a transitory component yields over-fitting. Indeed, the likelihood function does not increase sufficiently to offset the two additional parameters' penalty and therefore to increase the model's criterion.

Finally, we want to see how the stochastic drift of the permanent component and the transitory component are related to the historical dividend yield, which is a forward-looking measure of equity compensation. Table 3 reports the regression estimates from the regression of the log dividend yield dy_t (i.e. the log dividend-price ratio) on either $\widehat{x}_t$ or $\widehat{z}_t$ or both. While both $\widehat{x}_t$ and $\widehat{z}_t$ appear to be significant in the univariate regressions, the stochastic drift only explains a negligible fraction of dividend yield variation (about 1%) and is not significant in the bivariate regression. Instead, the adjusted-R^2 for the univariate regression with $\widehat{z}_t$ as independent variable is about 37% and does not increases in the bivariate regression. This suggests that the transitory component captures to a large extent the dynamics of the equity premium expected by the investors. A negative transitory shock (low $\widehat{z}_t$) leads with a very high probability to a higher compensation (high dy_t), whereas the link between a stochastic drift and the dividend yield is much weaker. Such a result is not surprising in light of the results in Figure 3 and Table 1. Dividend yield variation is indeed closely related with long-horizon variance ratios, which capture the importance of transitory risk relative to permanent risk.

Table 3 also reports the regression estimates when $\widehat{x}_t$ and $\widehat{z}_t$ are estimated in presence of stochastic volatility. The pattern is similar but the explanatory power of $\widehat{z}_t$ is even larger (the adjusted-R^2 is about 65%). This is likely due to the fact that the filtered variance v_t captures the unpriced variation of returns and therefore the resulting dynamics for $\widehat{z}_t$ represent a cleaner measure of equity compensation.

Figure 4: Time Series of the Stochastic Drift and Transitory Component of S&P 500 Returns.
This figure plots the stochastic drift $\widehat{x}_t$ and the transitory component $\widehat{z}_t$ of S&P 500 returns for the PT-CV model that assumes constant stock return variance. Data are at the monthly frequency from 02/1871 to 02/2016.

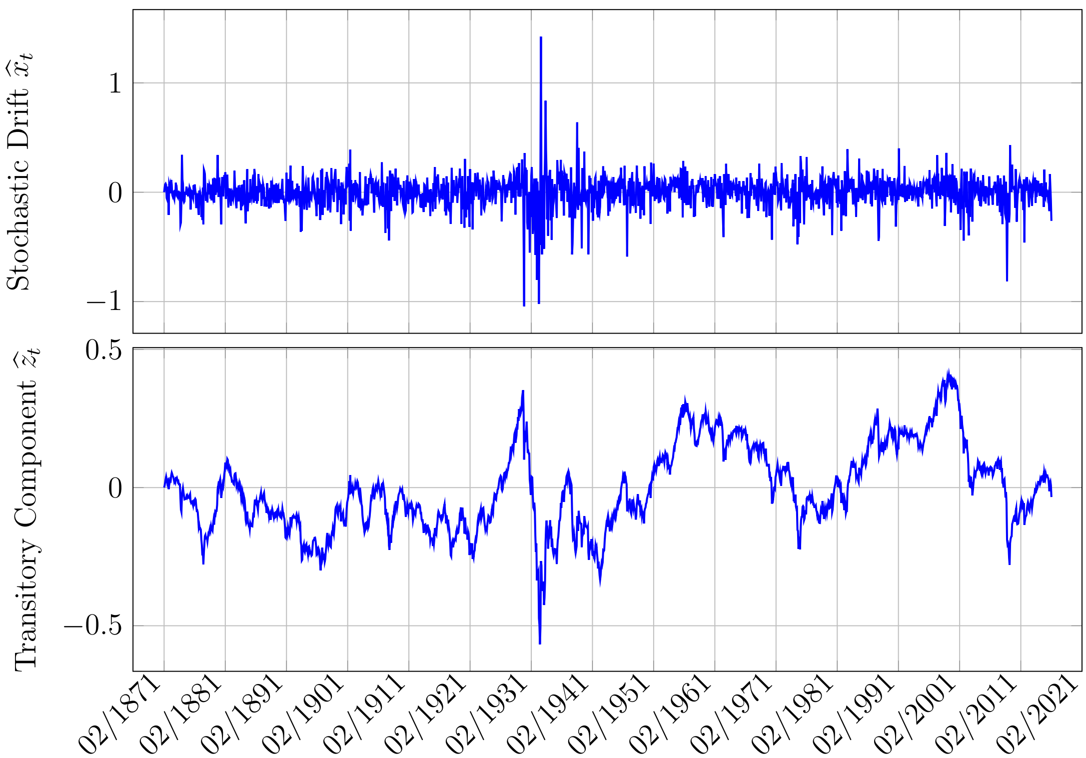


Figure 5 reports the scatter plots of the log dividend yield as a function of respectively $\widehat{x}_t$ and $\widehat{z}_t$ in presence of stochastic volatility: while in the former case we do not observe a clear linear relation, in the latter case we observe a strong negative slope in accord with the above regression results.

These results are robust to extreme observations (realizations of $\widehat{x}_t$ and $\widehat{z}_t$ outside $\pm$ 2 standard deviations around the average only have a negligible effect on the regression estimates), to sub-samples (the larger explanatory power of $\widehat{z}_t$ relative to $\widehat{x}_t$ obtains in both pre-war and post-war data; in particular, the full-sample correlation between the log dividend yield and $\widehat{z}_t$ is about -81% and increases beyond -95% in post-war data), as well as to time aggregation (averaging $dy_t, \widehat{x}_t$ and $\widehat{z}_t$ at yearly frequency and running the same regressions leads to very similar estimates).

To better understand the importance of correctly estimating the transitory component $\widehat{z}_t$, it is convenient to plot the time series of $\widehat{z}_t$ and (minus) the log dividend yield, that is

Table 3: Stochastic Drift, Transitory Shock, and the Dividend Yield.
This table reports the estimates from the regression of the log dividend yield $\log dy$ on $\widehat{x}$ and $\widehat{z}$:

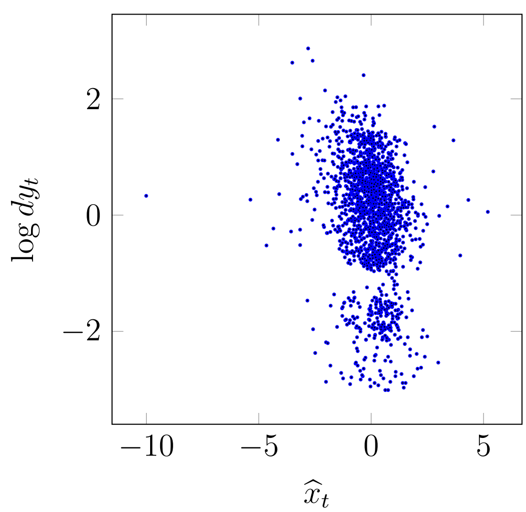
$$\log dy_t = a + b_x \widehat{x}_t + b_z \widehat{z}_t + \epsilon_t,$$

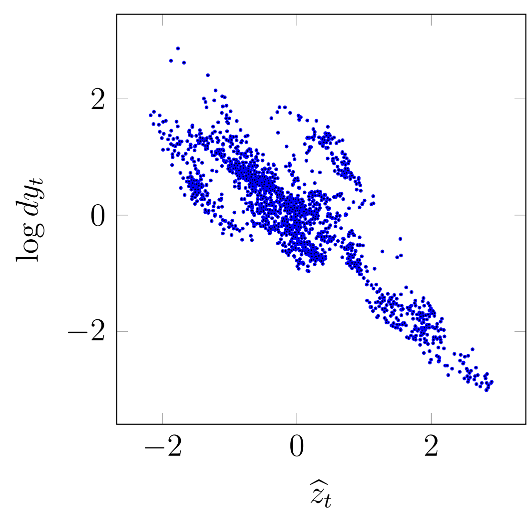
where all quantities are monthly and the regressions are run in case of either constant or stochastic volatility. Newey-West t-statistics are reported in parentheses and 10%, 5%, and 1% significance levels are denoted with *, **, and ***, respectively.

	constant volatility			stochastic volatility		
	(1)	(2)	(3)	(4)	(5)	(6)
$\widehat{x}$	-0.353***		-0.040	-0.830***		-0.221***
t-stat	(-4.16)		(-0.60)	(-5.57)		(-2.72)
$\widehat{z}$		-1.710***	-1.704***		-0.549***	-0.540***
t-stat		(-11.56)	(-11.27)		(-20.56)	(-19.53)
adj-R^2	0.01	0.37	0.37	0.05	0.65	0.66

Figure 5: Stochastic Drift, Transitory Shock, and the Dividend Yield.

The left and right panels show the scatter plots of the (standardized) dividend yield as a function of respectively the (standardized) $\widehat{x}$ and $\widehat{z}$ estimated in presence of stochastic volatility.

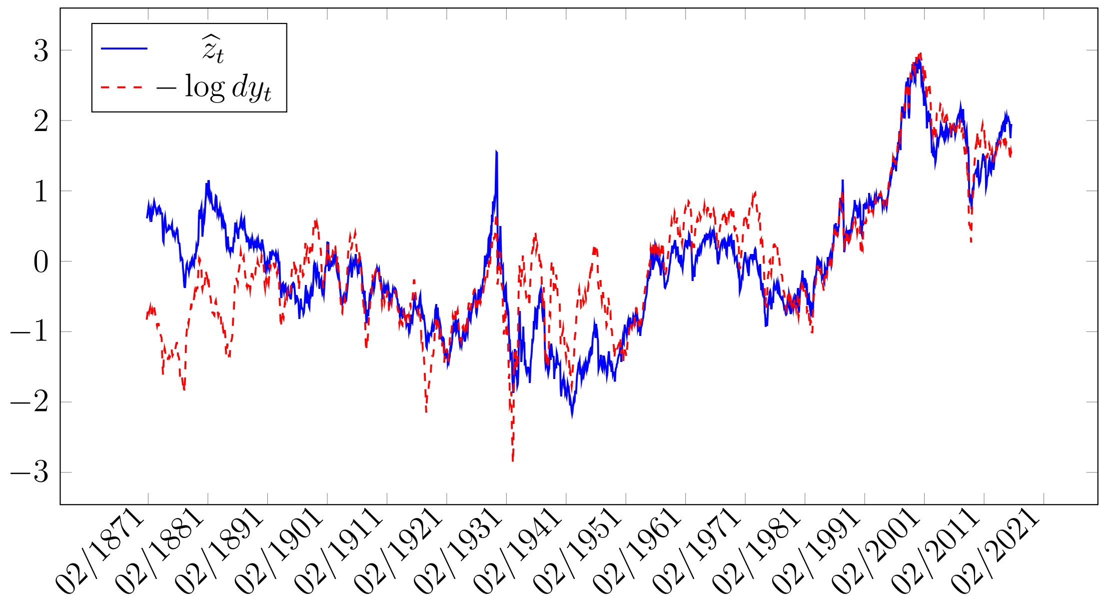




the usual valuation ratio of the stock market. Figure 6 shows that $\widehat{z}_t$ essentially matches all of the cyclical fluctuations of dy_t in the 20th Century. A model of return dynamics which disregards the estimation of a transitory component fails to capture the time-variation of expected returns (i.e. the dividend yield dynamics) and, hence, cannot correctly describe risk-adjusted returns.

Figure 6: Dividend Yield and Transitory Risk in the 20th Century.

Time-series of the (standardized) minus log dividend yield (i.e. the log price-dividend ratio) and the (standardized) transitory component $\widehat{z}$ estimated in presence of stochastic volatility.



5 Results

In this section, we first show that the accounting for a transitory component helps generate the observed term structures of equity risk. Second, we discuss the impact of the transitory component on the optimal portfolio weight. Third, we compare the performance of the different investment strategies and show that considering the transitory component of stock returns implies surges in the investor's portfolio returns.

5.1 Model-Implied Term Structure of Equity Risk

Prior to investigating the timing of equity risk in our model we start off with clarifying the definitions. The term structure of risk over horizon τ can be determined by studying the annualized variance of multi-period returns

$$\sigma_{S,t}(\tau) = \sqrt{\frac{1}{\tau}\text{Var}_t\left[\log\left(S_{t+\tau}/S_t\right)\right]}. \tag{20}$$

These annualized moments capture how the risk profile of the stock changes as the holding period increases. Recall that if the log price is a random walk with drift, the term structure implied by $\sigma_{S,t}(\tau)$ is flat because the variance of multi-period returns is a linear function of the horizon.

The literature (Belo et al., 2015; Marfè, 2015a) has also been investigating an alternative definition, namely,

$$\sigma_{S,t}(\tau) = \sqrt{\frac{1}{\tau} \log \left(\frac{\mathrm{E}_t[(S_{t+\tau}/S_t)^2]}{(\mathrm{E}_t[S_{t+\tau}/S_t])^2} \right)}, \tag{21}$$

which can be viewed as a first order approximation of (20), and thus a special case of the delta method.[8] In the remainder of this section we report term structure of equity risk using (21), as it can be solved for analytically in our setting. Unless stated otherwise, the expectations are taken with respect to the investor's observable filtration $\mathbf{F}^S$.

To demonstrate how well the different models under consideration are able to reproduce the empirically observed timing of equity risk discussed in Section 2, we depict in Figure 7 the term structure of volatility in the models with and without the transitory component along with the observed volatilities of nominal S&P 500 returns. We can clearly see that adding the transitory component helps to generate the observed downward-sloping pattern of equity risk. In line with empirical evidence, the model with both permanent and transitory components (and stochastic volatility) produces 1-year annualized standard deviation of returns of about 18% with a decline to about 15% at the 10-year horizon. In contrast, models that ignore the transitory component imply slightly upward-sloping term structures of equity risk, and do therefore not accurately replicate the observed timing of risk.

To provide further evidence on the importance of considering a transitory component to accurately model the timing of equity risk, we compute the variance ratios of stock returns. Variance ratios tells us whether variances of long horizon returns are proportional to the variance of a reference short horizon return. We use the 1-year horizon as a reference and compute variance ratios as

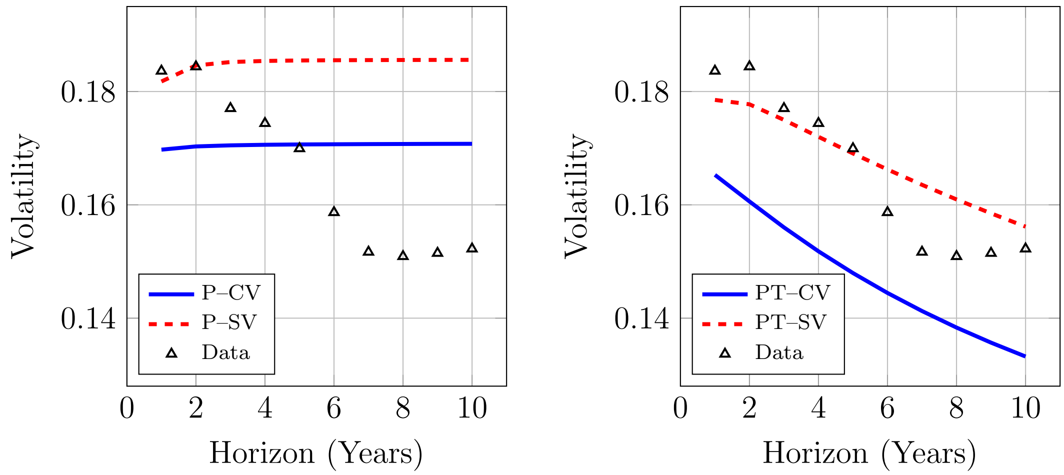
$$\mathrm{VR}_t(\tau) = \frac{\sigma_{S,t}(\tau)^2}{\sigma_{S,t}(1)^2}.$$

Intuitively, variance ratios above (below) one imply concentration of risk at longer (shorter) horizons.

Figure 8 depicts the term structure of variance ratios in the models with and without the transitory component together with the observed variance ratios of nominal S&P 500 returns.

[8] By an application of the delta method we have that $\mathrm{Var}\left[f(X)\right] \approx \left(f'\left(\mathrm{E}[X]\right)\right)^2 \mathrm{Var}[\mathrm{X}]$, and hence $\mathrm{Var}\left[\log(X)\right] \approx \frac{1}{(\mathrm{E}[X])^2}\mathrm{Var}[\mathrm{X}] = \frac{\mathrm{E}[X^2]}{(\mathrm{E}[X])^2} - 1$. Finally, using $\log(1+x) \approx x$ we obtain (21).

Figure 7: Term Structure of Volatility.
The left panel shows the term structure of volatility for horizons ranging from 1 to 10 years in the model with permanent component with stochastic drift (P). The right panel depicts the same term structure in the model with both permanent component with stochastic drift and transitory component (PT). CV stands for constant volatility model and SV stands for stochastic volatility model. State variables are set to their long-term means and the parameter values are reported in Table 2. Data markers display the volatilities (up to the 10-year horizon) of nominal S&P 500 yearly returns from the full sample (1872–2015).

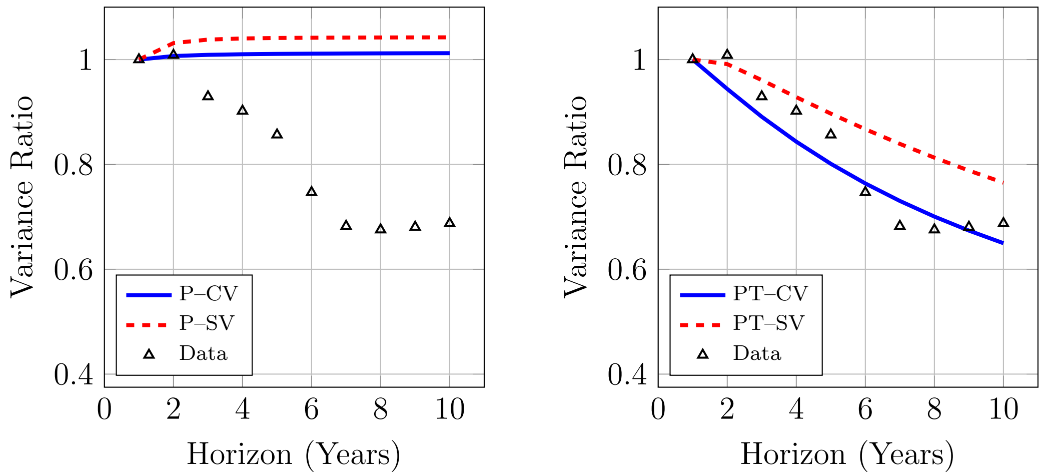


Once again, the main noteworthy result is that the models with both a permanent component and a transitory component produce a downward-sloping term structure of variance ratios (see right panel). The magnitude of model-implied variance ratios is in line with its empirical counterpart. For example, we see that a 10-year variance ratio in the model with permanent and transitory components is around 0.7, matching the 0.687 observed in the data. The variance ratios in the model with only a permanent component, on the contrary, are all above one and approximately flat (see left panel). Proposition 4 shows that this conclusion is not merely attributed to the set of parameters we estimate, but is in fact a property of the model when the transitory component is turned off ($z_t \equiv 0$). That is, the stock price dynamics without transitory component are unable to produce a downward-sloping term structure of equity risk.

Proposition 4. *When $z_t \equiv 0$ and $v_t \equiv \overline{v}$, the true variance of returns at a τ-year horizon is*

$$\sigma_S(t,\tau)^2 = \overline{v} + \frac{\sigma_x^2}{\kappa_x^2} - \frac{1}{\tau}\frac{e^{-2\kappa_x\tau}\left(1+3e^{2\kappa_x\tau}-4e^{\kappa_x\tau}\right)\sigma_x^2}{2\kappa_x^3}, \tag{22}$$

Figure 8: Term Structure of Variance Ratios.
The left panel depicts the term structure of variance ratios for horizons ranging from 1 to 10 years in the model with permanent component with stochastic drift (P). The right panel depicts the same term structure in the model with both permanent component with stochastic drift and transitory component (PT). CV stands for constant volatility model and SV stands for stochastic volatility model. State variables are set to their long-term means and the parameter values are reported in Table 2. Data markers display the variance ratios (up to the 10-year horizon) of nominal S&P 500 yearly returns from the full sample (1872–2015).



the filtered variance of returns at a τ-year horizon is

$$\sigma_S(t,\tau)^2 = \frac{(\overline{\gamma}_x + \overline{v}\kappa_x)^2}{\overline{v}\kappa_x^2} - \frac{1}{\tau}\frac{e^{-2\kappa_x\tau}(1-e^{\kappa_x\tau})((1-3e^{\kappa_x\tau})\overline{\gamma}_x - 4e^{\kappa_x\tau}\overline{v}\kappa_x)\overline{\gamma}_x}{2\overline{v}\kappa_x^3} \tag{23}$$

and in both cases we have

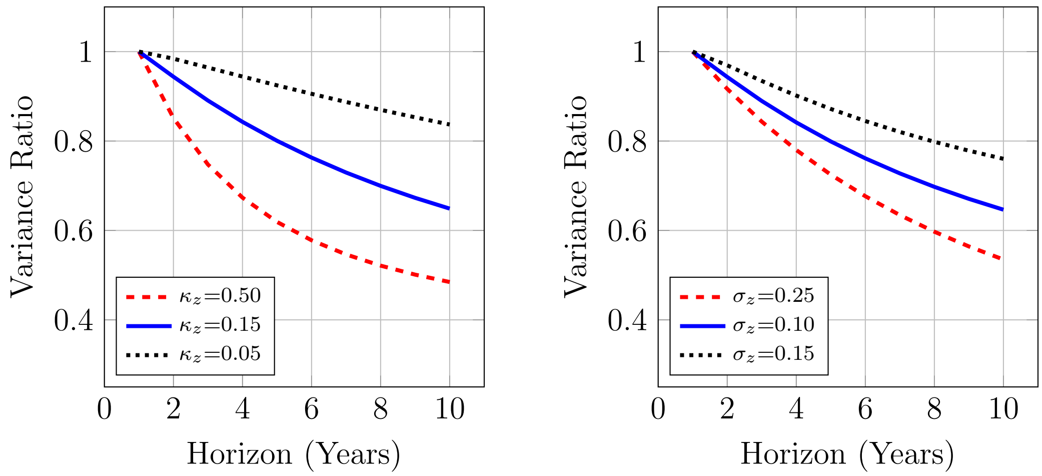
$$\partial_\tau \sigma_S(t,\tau)^2 \geq 0$$

for all $\kappa_x \geq 0$.

Proof. See Appendix A.3.

This result states that the slope of the term structure of risk in the model without the transitory components is nonnegative at any horizon. Proposition 4 covers the case of constant volatility, but from Figures 7 and 8 we see that with stochastic volatility the term structure of risk is even more upward sloping. Intuitively and similarly to the time-varying drift of the permanent component x_t, stochastic volatility increases the uncertainty of stock returns in the long run because these shocks accumulate (see Equation (1)).

Figure 9: Effect of the Mean-Reversion Speed and the Volatility of the Transitory Component on the Timing of Equity Risk.
The left panel shows how the variance ratio of stock returns changes as the mean-reversion speed of the transitory component κ_z varies. The right panel shows how the variance ratio of stock returns changes as the volatility of the transitory component σ_z varies. Both panels consider a constant volatility model. State variables are set to their long-term means and the parameter values are reported in Table 2.



Having established the fact that adding the transitory component to the model of stock returns enables us to reconcile the model-implied timing of risk with empirical evidence, we now turn to comparative statics results. Figure 9 illustrates the effect of varying the parameters the transitory component, namely the mean-reversion speed κ_z, and the volatility σ_z. The left panel shows that a higher mean-reversion speed decreases the slope of the variance ratios. Since κ_z governs the speed at which transitory shocks die out, the larger κ_z the higher the risk in the short run relative to the long run. When the transitory component becomes more persistent (i.e. lower κ_z), short-term and long-term risks align and therefore the slope of the term structure of variance ratios flattens out. The right panel of Figure 9 shows that varying the volatility of the transitory component σ_z has a similar effect as varying its mean-reversion speed κ_z. Indeed, a more volatile transitory component implies a more downward-sloping term structure of variance ratios because increasing σ_z increases short-term equity risk but has no influence on long-term equity risk.

5.2 Optimal Portfolio and Hedging Demands

Table 4 provides the decomposition of the investor's portfolio for different coefficients of relative risk aversion (5, 7, and 10) and investment horizons (1 month, 1 year, and 5 years). The optimal proportion of wealth allocated to the stock, the myopic stock allocation, and the different components of hedging demands are reported using the long-term mean values of the state variables.

Let us first analyze the signs of the portfolio components. The myopic demand is positive as long as the expected excess return on the stock is positive. The signs of the hedging demands depend on the correlation between stock returns and the state variables, as well as on the sensitivity of the marginal utility to changes in the state variables. To hedge shocks to the transitory component investors tend to increase their risky investment, while to hedge the stochastic drift investors decrease their risky investment. The opposite signs of the hedging demands are due to the opposite effects the two shocks have on expected returns. Since the transitory component is positively correlated with realized returns and impacts negatively expected returns, the stock is a good hedging instrument against a deterioration in investment opportunities and therefore this hedging demand is positive. On the other hand, positive shocks to the permanent component increase expected returns and coincide with positive realized returns. This makes the stock a bad hedging instrument and therefore implies a negative hedging demand. Similarly to the stochastic drift, stochastic volatility gives rise to a negative hedging demand. This is because the negative correlation between volatility and realized returns implies that volatility is high when returns are low.

Turning our attention to the relative importance of the different components of the risky share, we note that the magnitude of the hedging demands is smaller than that of the myopic stock allocation, but is not negligible. For example, in the constant volatility model with a 5-year horizon, the contribution of the stock allocation to hedge the transitory shock amounts to around 30% of the size of the myopic demand.

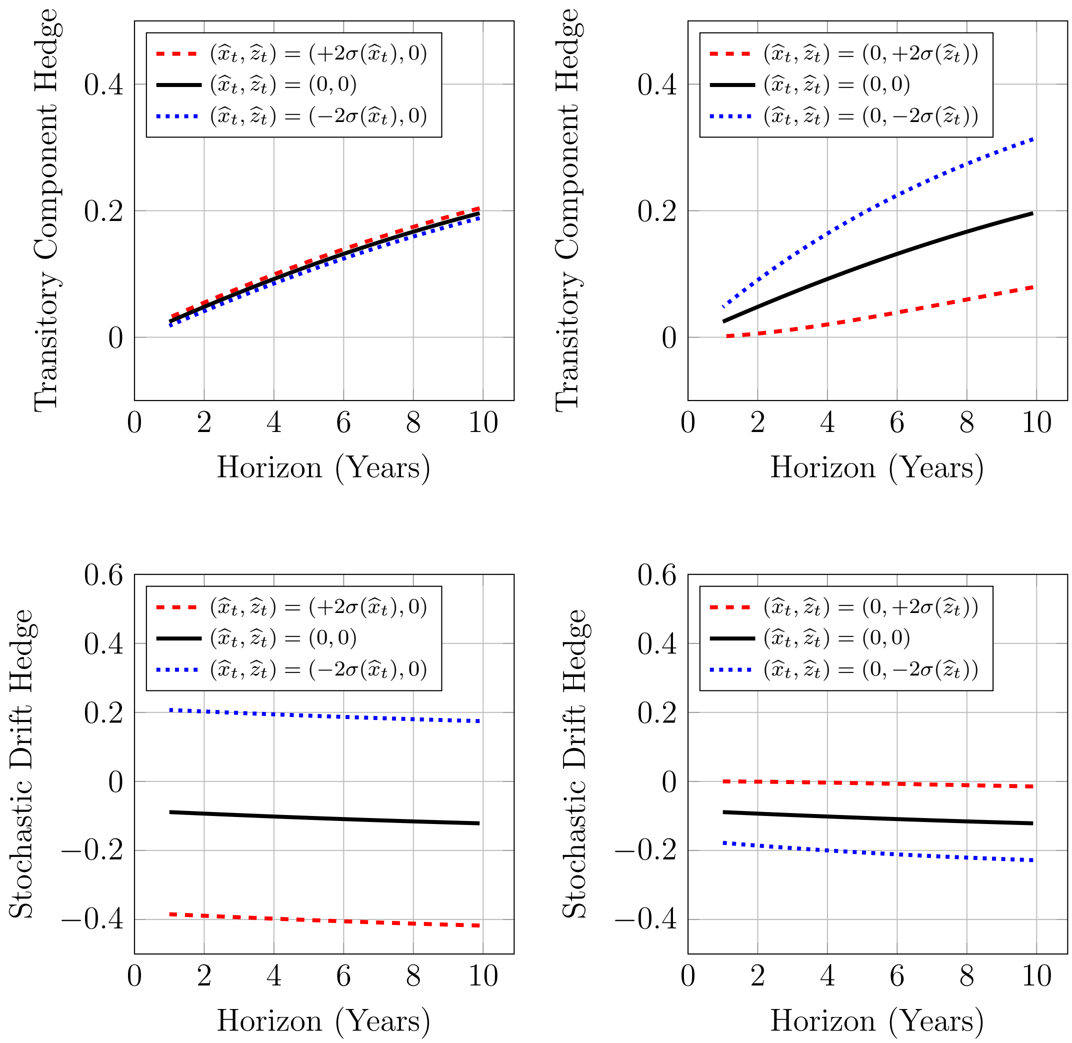
As the investment horizon increases, the magnitude of the hedging demands tends to increase. The horizon effect is particularly pronounced for the transitory shock hedging demand and for the volatility hedging demand. Indeed, the magnitude of these hedging demands increases tenfold as the investment horizon increases from 1 month to 1 year. The impact of the horizon on the magnitude of the stochastic drift hedging demand, however, is weaker. In the constant volatility model, the strong positive relation between the horizon and the transitory shock hedging demand dominates the weak negative relation between the horizon and the stochastic drift hedging demand, resulting in an overall positive horizon effect. That is, investors with longer horizon tend to invest more in the stock. In the stochastic volatility model, the strong negative relation between the horizon and the volatility hedging demand dominates. Therefore, investors with longer horizons choose a lower risky share.

We further investigate the signs, magnitudes, and long horizon patterns of the hedging demands as the state variables deviate from their long-term means. Figure 10 illustrates

Table 4: Decomposition of Optimal Stock Allocation.
This table presents the optimal stock allocation, the myopic allocation, and the hedging demands for the two cases: the model with constant volatility and the model with stochastic volatility. We consider investors with coefficients of relative risk aversion γ of 5, 7, and 10, and investment horizons $T - t$ of 1 month, 1 year and 5 years. State variables are set to their long-term means and the parameter values are reported in Table 2.

	Constant volatility model			Stochastic volatility model		
γ	1m	1y	5y	1m	1y	5y
	Mean optimal allocation to stocks					
5	0.4048	0.4220	0.5104	0.3475	0.2822	0.2788
7	0.2853	0.2981	0.3645	0.2455	0.1976	0.1948
10	0.1978	0.2070	0.2551	0.1705	0.1364	0.1342
	Myopic stock allocation					
5	0.5017	0.5017	0.5017	0.4121	0.4121	0.4121
7	0.3583	0.3583	0.3583	0.2944	0.2944	0.2944
10	0.2508	0.2508	0.2508	0.2061	0.2061	0.2061
	Stock allocation to hedge the stochastic drift					
5	-0.0998	-0.1103	-0.1309	-0.0609	-0.0870	-0.0844
7	-0.0752	-0.0831	-0.0996	-0.0460	-0.0644	-0.0623
10	-0.0546	-0.0604	-0.0729	-0.0335	-0.0463	-0.0446
	Stock allocation to hedge the transitory shock					
5	0.0030	0.0306	0.1397	0.0014	0.0100	0.0387
7	0.0022	0.0229	0.1058	0.0011	0.0074	0.0284
10	0.0016	0.0166	0.0772	0.0008	0.0053	0.0202
	Stock allocation to hedge volatility					
5				-0.0052	-0.0529	-0.0877
7				-0.0039	-0.0397	-0.0657
10				-0.0028	-0.0287	-0.0475

Figure 10: Effect of the Transitory Component and the Drift of the Permanent Component on Hedging Demands.
The left panels show the hedging demands as a function of the investment horizon for different values of the drift of the permanent component $\widehat{x}_t$. The curves correspond to $\widehat{x}_t = 0, \pm 2\,\sigma\left(\widehat{x}_t\right)$, where $\sigma\left(\widehat{x}_t\right)$ is the long-term standard deviation of the stochastic drift. The right panels show the hedging demands as a function of the investment horizon for different values of the transitory component $\widehat{z}_t$. The curves correspond to $\widehat{z}_t = 0, \pm 2\,\sigma\left(\widehat{z}_t\right)$, where $\sigma\left(\widehat{z}_t\right)$ is the long-term standard deviation of the transitory component. If not stated otherwise, state variables are set to their long-term means and the parameter values are reported in Table 2. The numbers reported are based on the constant volatility model.



the effect of varying the stochastic drift of the permanent component (left panels) and the transitory component (right panels) in the constant volatility model. We observe that the shape of the hedging demands remain unchanged when varying the aforementioned state variables. Indeed, the transitory component hedging demand remains positive and increases with the horizon. The stochastic drift hedging demand depends weakly on the horizon and remains negative with an exception occurring when $\widehat{x}_t$ is two standard deviations below its long-term mean. In this case, the expected excess return is negative, the investor shorts the stock, and therefore the sign of the hedging demand is reversed. This effect is temporary and quickly disappears because $\widehat{x}_t$ reverts back at a high speed towards its long-term mean.

5.3 In-sample Portfolio Performance

In this section, we compare the in-sample performance of the different investment strategies. That is, we use the parameters estimated over the entire sample (see Table 2) to implement strategies that invest in the S&P 500 and in a riskless asset with constant return. In-sample, strategies considering the transitory component of stock returns significantly outperform those that ignore it.

Consider an investor with an investment horizon equal to T and an initial wealth equal to \$1. The investor starts implementing her strategy at time $t = 0$ and rebalances every month. When reaching the horizon $t = T$, the investors reiterates the process until reaching the terminal date of the sample. Since data are at the monthly frequency, monthly portfolio excess returns are provided over the entire sample.

Table 5 provides the moments of the portfolio excess returns and the terminal wealth associated to each model for different values of the investment horizon T and risk aversion γ. Table 6 reports the corresponding decomposition of volatility and kurtosis of portfolio returns. Following Segal et al. (2015), annualized good and bad volatility, GV and BV, are measured as follows:

$$GV = \sqrt{\frac{\Delta}{N}\sum_{i=1}^{N}\mathbf{1}_{\left\{r^p_{i\Delta}\geq \bar{r}^p\right\}}\left(r^p_{i\Delta}-\bar{r}^p\right)^2}, \qquad BV = \sqrt{\frac{\Delta}{N}\sum_{i=1}^{N}\mathbf{1}_{\left\{r^p_{i\Delta}< \bar{r}^p\right\}}\left(r^p_{i\Delta}-\bar{r}^p\right)^2},$$

where r^p is the portfolio return and $\bar{r}^p$ is the sample average of portfolio returns. Similarly, good and bad kurtosis, GK and BK are computed as follows:

$$GK = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{\left\{r^p_{i\Delta}\geq \bar{r}^p\right\}}\left(r^p_{i\Delta}-\bar{r}^p\right)^4}}{\mathrm{Var}(r^p)^2}, \qquad BK = \frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{\left\{r^p_{i\Delta}< \bar{r}^p\right\}}\left(r^p_{i\Delta}-\bar{r}^p\right)^4}}{\mathrm{Var}(r^p)^2},$$

where $\mathrm{Var}(r^p)$ is the sample variance of portfolio returns.

Table 5 shows that considering a stochastic drift (P-CV) increases the mean, volatility, Sharpe ratio, skewness, and kurtosis of portfolio returns compared to the model with constant mean and constant volatility (CM-CV). Because the increase in the mean is large, terminal wealth is several order of magnitude larger when expected stock returns are assumed to be stochastic.

In contrast, considering stochastic return variance (P-SV) decreases the mean and Sharpe ratio of portfolio returns but increases their volatility, skewness, and kurtosis (see Table 5). Although both volatility and kurtosis increase importantly, it is in fact good volatility and good kurtosis that increase, while bad volatility and bad kurtosis remain at levels observed under constant stock return variance (see Table 6). That is, the investor enjoys the increase in return volatility, larger return spikes because of higher skewness, a larger number of return spikes because of higher good kurtosis when considering stochastic stock return variance.

When the investor accounts for the presence of the transitory component (PT-CV, PT-SV), the mean, volatility, skewness, and kurtosis of her portfolio returns increase (see Table 5). Since increases in mean and volatility are of similar magnitudes, the Sharpe ratio remains the same. Importantly, the increase in volatility does in fact not hurt the investor because the increase in good volatility is larger than that in bad volatility (see Table 6). Moreover, the increase in kurtosis is purely beneficial to the investor because good kurtosis increases and bad kurtosis decreases. That is, the investor enjoys higher returns, larger return spikes, a larger number of return spikes, a lower number of extreme negative return, and is not hurt by the increase in volatility when accounting for the transitory component of stock returns.

These results show that accounting for the transitory component of stock returns has no significant impact on the left tail of the distribution of portfolio returns and mainly impacts its right tail through higher good volatility, skewness, and good kurtosis. Properly modeling the behavior of short-term equity returns implies surges in portfolio returns, which are captured by measures of performance such as good volatility, skewness, and good kurtosis. This shows that modeling the transitory component of stock returns is beneficial to the investor, and even more so when return variance is considered to be stochastic. Furthermore, it is worth noting that these benefits are not captured by the most common measure of portfolio performance, namely the Sharpe ratio.

Tables 5 and 6 show that the portfolio performance benefits of considering the transitory component are robust to changes in investment horizon and risk aversion. An increase in the investment horizon has a weak impact on the mean, volatility, and Sharpe ratio, but it increases skewness and good kurtosis across all strategies. An increase in risk aversion has no impact on the Sharpe ratio because it decreases the mean and volatility by the same percentage across all strategies. Furthermore, an increase in risk aversion impacts skewness and kurtosis only when return volatility is stochastic. In this case, an increase in risk aversion increases skewness and good kurtosis, while it decreases bad kurtosis.

To further quantify the relative performance of the different strategies without directly

Table 5: In-Sample Portfolio Moments vs. Investment Horizon and Risk Aversion.

Mean, volatility, and Sharpe ratio are in annualized terms. In columns 1 to 3, the investment horizon is set to 1 year. In columns 4 to 6, risk aversion is $\gamma = 5$ and the investment horizon is set to 1 month, 1 year, and 5 years, respectively. P, PT, CM, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant mean, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1871 to 02/2016.

	Risk Aversion			Horizon		
	5	7	10	1m	1y	5y
			Mean			
CM-CV	2.11%	1.51%	1.05%	2.11%	2.11%	2.11%
P-CV	21.93%	15.57%	10.85%	21.95%	21.93%	21.93%
PT-CV	22.31%	15.83%	11.03%	22.29%	22.31%	22.40%
P-SV	17.87%	12.67%	8.82%	18.49%	17.87%	17.70%
PT-SV	19.34%	13.71%	9.55%	19.94%	19.34%	19.35%
			Volatility			
CM-CV	6.52%	4.65%	3.26%	6.52%	6.52%	6.52%
P-CV	34.81%	24.72%	17.23%	34.83%	34.81%	34.81%
PT-CV	36.66%	26.03%	18.14%	36.62%	36.66%	37.01%
P-SV	42.59%	30.35%	21.20%	43.10%	42.59%	42.65%
PT-SV	46.41%	33.08%	23.12%	46.81%	46.41%	46.61%
			Sharpe Ratio			
CM-CV	0.32	0.32	0.32	0.32	0.32	0.32
P-CV	0.63	0.63	0.63	0.63	0.63	0.63
PT-CV	0.61	0.61	0.61	0.61	0.61	0.61
P-SV	0.42	0.42	0.42	0.43	0.42	0.42
PT-SV	0.42	0.41	0.41	0.43	0.42	0.42
			Skewness			
CM-CV	0.55	0.55	0.55	0.55	0.55	0.55
P-CV	9.10	9.10	9.10	9.10	9.10	9.10
PT-CV	9.73	9.73	9.72	9.72	9.73	9.87
P-SV	25.68	25.86	25.99	24.85	25.68	25.84
PT-SV	26.84	27.03	27.16	26.02	26.84	26.71
			Kurtosis			
CM-CV	20.71	20.71	20.71	20.71	20.71	20.71
P-CV	124.24	124.20	124.16	124.33	124.24	124.25
PT-CV	141.47	141.32	141.21	141.20	141.47	146.44
P-SV	881.43	890.23	896.66	841.50	881.43	889.25
PT-SV	942.14	951.18	957.76	901.42	942.14	935.33
			Terminal Wealth			
CM-CV	2.2×10^4	1.0×10^4	5.9×10^3	2.2×10^4	2.2×10^4	2.2×10^4
P-CV	1.7×10^{14}	2.8×10^{11}	1.5×10^9	1.7×10^{14}	1.7×10^{14}	1.7×10^{14}
PT-CV	1.8×10^{14}	3.2×10^{11}	1.7×10^9	1.8×10^{14}	1.8×10^{14}	1.9×10^{14}
P-SV	4.8×10^{11}	3.9×10^9	7.2×10^7	8.7×10^{11}	4.8×10^{11}	3.9×10^{11}
PT-SV	2.0×10^{12}	1.1×10^{10}	1.6×10^8	3.5×10^{12}	2.0×10^{12}	1.9×10^{12}

Table 6: In-Sample Portfolio Return Volatility and Kurtosis Decomposition vs. Investment Horizon and Risk Aversion.

Volatility is in annualized terms. In columns 1 to 3, the investment horizon is set to 1 year. In columns 4 to 6, risk aversion is $\gamma = 5$ and the investment horizon is set to 1 month, 1 year, and 5 years, respectively. P, PT, CM, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant mean, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1871 to 02/2016.

	Risk Aversion			Horizon		
	5	7	10	1m	1y	5y
			Good Volatility			
CM-CV	4.50%	3.22%	2.25%	4.50%	4.50%	4.50%
P-CV	32.37%	22.99%	16.02%	32.39%	32.37%	32.37%
PT-CV	34.23%	24.30%	16.93%	34.19%	34.23%	34.54%
P-SV	40.64%	28.97%	20.25%	40.98%	40.64%	40.72%
PT-SV	44.40%	31.67%	22.15%	44.64%	44.40%	44.57%
			Bad Volatility			
CM-CV	4.71%	3.37%	2.36%	4.71%	4.71%	4.71%
P-CV	12.80%	9.09%	6.33%	12.82%	12.80%	12.80%
PT-CV	13.13%	9.32%	6.50%	13.12%	13.13%	13.30%
P-SV	12.76%	9.03%	6.28%	13.36%	12.76%	12.69%
PT-SV	13.50%	9.55%	6.64%	14.09%	13.50%	13.64%
			Good Kurtosis			
CM-CV	16.09	16.09	16.09	16.09	16.09	16.09
P-CV	123.75	123.70	123.67	123.84	123.75	123.76
PT-CV	141.00	140.85	140.74	140.73	141.00	145.96
P-SV	880.82	889.64	896.09	840.77	880.82	888.66
PT-SV	941.67	950.73	957.32	900.89	941.67	934.86
			Bad Kurtosis			
CM-CV	4.62	4.62	4.62	4.62	4.62	4.62
P-CV	0.50	0.49	0.49	0.50	0.50	0.49
PT-CV	0.47	0.47	0.47	0.47	0.47	0.49
P-SV	0.61	0.59	0.57	0.73	0.61	0.60
PT-SV	0.47	0.45	0.44	0.54	0.47	0.47

relying on portfolio return moments, we follow Johannes et al. (2014) and define the certainty equivalent return, $CER_{m,t}$, of model m at time t, as follows:

$$J(t, X_{m,t}; m) \equiv e^{-\delta t} \frac{\left(W_t e^{(T-t)CER_{m,t}}\right)^{1-\gamma}}{1-\gamma},$$
$$m \in \{\text{CM-CV}, \text{P-CV}, \text{PT-CV}, \text{P-SV}, \text{PT-SV}\},$$

where $J(.,.;m)$ and $X_{m,t}$ are the value function and the vector of state variables associated to model m, respectively. That is, the certainty equivalent return is defined as the yield of a fictitious riskless asset that makes the investor indifferent between implementing her optimal risky strategy and investing her entire wealth in this riskless asset.

Table 7 confirms our previous statement that modeling the transitory component of equity returns is particularly beneficial to the investor when return volatility is stochastic. Indeed, the certainty equivalent return obtained by considering the transitory component is about 1% and 10% larger than that obtained by ignoring it when return volatility is constant and stochastic, respectively. Since the fraction of wealth invested in the stock increases with the horizon and decreases with risk aversion for all investment strategies (see Table 4), all strategies converge to riskless strategies when the investment horizon decreases and when risk aversion increases. For this reason, the ratio of certainty equivalent returns increases with the horizon and decreases with risk aversion.

Table 7: In-Sample Ratio of Certainty Equivalent Returns.
CER stands for certainty equivalent return. In columns 1 to 3, the investment horizon is set to 1 year. In columns 4 to 6, risk aversion is $\gamma = 5$ and the investment horizon is set to 1 month, 1 year, and 5 years, respectively. P, PT, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1871 to 02/2016.

	Risk Aversion			Horizon		
	5	7	10	1m	1y	5y
			Ratio of Mean CER			
PT-CV/P-CV	1.0127	1.0110	1.0092	1.0126	1.0127	1.0195
PT-SV/P-SV	1.1037	1.0880	1.0717	1.0950	1.1037	1.1170
			Ratio of Median CER			
PT-CV/P-CV	1.0096	1.0084	1.0072	1.0030	1.0096	1.0170
PT-SV/P-SV	1.0927	1.0771	1.0618	1.0770	1.0927	1.0939

5.4 Out-of-Sample Portfolio Performance

To investigate the out-of-sample performance of the different strategies, we assume that the investor re-estimates the parameters of the models every year. More precisely, we assume that the investor starts by using data from 02/1871 to 01/1900 to estimate the first set of parameters. Assuming an investment horizon of one year, the investor trades and updates her beliefs about $\widehat{x}_t$ and $\widehat{z}_t$ from 02/1900 to 01/1901 by fixing the set of estimated parameters. The investor then uses data from 02/1871 to 01/1901 to estimate a second set of parameters that allows her to trade and update her beliefs from 02/1901 to 01/1902. This process is undertaken until reaching the terminal date 02/2016, and therefore provides a time series of out-of-sample monthly portfolio returns from 02/1900 to 02/2016.[9]

Since the investor re-estimates the parameters of the model every year, parameters are constant over twelve consecutive months and then change. To understand how the shape of the term structure of equity risk varies with economic conditions, we compute the median mean-reversion speed and volatility of the transitory component during NBER recessions and expansions. For the model with constant return volatility (PT-CV model), the median mean-reversion speed is 0.29 in recessions and 0.19 in expansions, whereas the median volatility is 0.18 in recessions and 0.18 in expansions. For the model with stochastic return volatility (PT-SV model), the median mean-reversion speed is 0.22 in recessions and 0.17 in expansions, whereas the median volatility is 0.18 in recessions and 0.18 in expansions. Since an increase in either the mean-reversion speed or the volatility of the transitory component decreases the slope of the term structure of variance ratio (Section 5.1), the slope of term structure of variance ratio is particularly negative in recessions. This result is consistent with the findings of van Binsbergen et al. (2013) and Aït-Sahalia et al. (2015), who show that the slope of the term structure of equity yields is pro-cyclical. Furthermore, this result suggests that accounting for the transitory component of stock returns should deliver more portfolio performance benefits in recessions than in expansions. We confirm this statement below.

Tables 8 and 9 provides the portfolio return moments of the different strategies. All learning-based strategies beat the constant mean and volatility strategy (CM-CV) in terms of Sharpe ratio, good relative to bad volatility, skewness, and good relative to bad kurtosis. That is, learning significantly improves portfolio performance in all dimensions. This result is particularly interesting given previous evidence that the out-of-sample predictive power and portfolio performance of a constant mean and volatility model is very hard to beat (Welch and Goyal, 2008; Johannes et al., 2014).

Adding a transitory component to the model increases skewness, good kurtosis, and

[9] As in Johannes et al. (2014), we prevent the occurrence of a few extreme portfolio positions typically obtained during periods of very low return volatility by bounding the portfolio weight at -7 and $+7$.

Table 8: Unconditional Out-of-Sample Portfolio Moments.
Mean, volatility, and Sharpe ratio are in annualized terms. The investment horizon is $T = 1$, risk aversion is $\gamma = 5$, and initial wealth is $W_0 = \$1$. P, PT, CM, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant mean, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1900 to 02/2016.

	Mean	Volatility	Sharpe Ratio	Skewness	Kurtosis	Terminal Wealth
CM-CV	1.19%	4.24%	0.28	-1.63	18.73	1,162
P-CV	24.38%	39.25%	0.62	7.58	101.88	8.0×10^{11}
PT-CV	23.28%	43.82%	0.53	9.53	153.15	8.5×10^{10}
P-SV	18.54%	31.05%	0.60	4.96	98.38	1.2×10^{9}
PT-SV	21.78%	35.92%	0.61	5.63	98.73	1.4×10^{10}

Table 9: Unconditional Out-of-Sample Portfolio Return Volatility and Kurtosis Decomposition.
Volatility is in annualized terms. The investment horizon is $T = 1$ and risk aversion is $\gamma = 5$. P, PT, CM, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant mean, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1900 to 02/2016.

	Good Volatility	Bad Volatility	Good Kurtosis	Bad Kurtosis
CM-CV	2.64%	3.32%	2.34	16.39
P-CV	35.63%	16.47%	100.54	1.33
PT-CV	40.35%	17.10%	151.94	1.21
P-SV	26.01%	16.96%	86.52	11.85
PT-SV	30.62%	18.77%	92.06	6.67

good volatility, decreases bad kurtosis, and has a weak impact on bad volatility irrespective of whether return volatility is constant or stochastic. The Sharpe ratio, however, decreases when return volatility is constant and is unaffected when it is stochastic. This shows that the benefits of adding a transitory component in the model are particularly important when return volatility is stochastic.

Surprisingly, considering stochastic volatility in the model with both permanent and transitory components (PT) has out-of-sample implications that are quite different from the in-sample implications. Indeed, while considering stochastic volatility tends to increase the Sharpe ratio and decrease both skewness and kurtosis out-of-sample, it decreases the Sharpe ratio and increases both skewness and kurtosis in-sample. This shows that in-sample portfolio performance analyses can be misleading and might therefore be completely uninformative on the performance of actual real-time portfolio strategies.

We have provided evidence that the observed slope of the term structure of variance ratios

Table 10: Out-of-Sample Portfolio Moments in NBER Recession and Expansion. Mean, volatility, and Sharpe ratio are in annualized terms. The investment horizon is $T = 1$ and risk aversion is $\gamma = 5$. P, PT, CM, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant mean, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1900 to 02/2016.

	Mean		Volatility		Sharpe Ratio		Skewness		Kurtosis	
	Exp.	Rec.	Exp.	Rec.	Exp.	Rec.	Exp.	Rec.	Exp.	Rec.
CM-CV	2.61%	-2.99%	3.35%	6.00%	0.78	-0.50	-0.35	-1.68	5.77	15.28
P-CV	16.72%	46.87%	25.44%	64.14%	0.66	0.73	6.18	5.46	108.37	46.41
PT-CV	14.46%	49.17%	27.23%	72.92%	0.53	0.67	6.88	6.85	122.85	69.16
P-SV	16.45%	24.70%	23.81%	46.16%	0.69	0.54	5.63	3.60	99.26	58.45
PT-SV	19.23%	29.29%	29.23%	50.69%	0.66	0.58	4.39	5.35	56.92	79.07

is negative unconditionally and decreases as economic conditions deteriorate (Section 2). In addition, the model-implied term structure of variance ratios can potentially be downward-sloping if the transitory component of returns is considered, whereas it can only be upward-sloping if the transitory component is ignored (Section 5.1). Therefore, the relative portfolio benefits of considering the transitory component as opposed to ignoring it should be larger in bad times than in good times. To investigate this matter, we compute the portfolio moments of each strategy conditional on being in NBER recession and expansion.

Tables 10 and 11 report the portfolio moments of each strategy conditional on being in NBER recession and expansion. As expected, the unconditional portfolio benefits obtained by considering a transitory component mostly come from the significantly better performance observed in NBER recession. Indeed, the relative increase in skewness, good kurtosis, and good volatility obtained by considering the transitory component is larger in recession than in expansion. In addition, the transitory component decreases bad volatility and bad kurtosis in recession but not necessarily in expansion. When volatility is stochastic, the transitory component increases the Sharpe ratio in recession more than it decreases it in expansion, lending further support that the benefits of the transitory component concentrate in bad times.

Table 12 provides the ratio of certainty equivalent returns obtained by considering and ignoring the transitory component of stock returns. Adding a transitory component to the model tends to increase the certainty equivalent return, especially so when stock return volatility is stochastic. Indeed, the mean and median certainty equivalent return increase by about 60% and 20%, respectively, when volatility is stochastic. When return volatility is constant, however, the mean and median certainty equivalent returns increase and decrease by 2%, respectively. Since return volatility has been documented to be indeed time-varying

Table 11: Out-of-Sample Portfolio Return Volatility and Kurtosis Decomposition in NBER Recession and Expansion.
Volatility is in annualized terms. The investment horizon is $T = 1$ and risk aversion is $\gamma = 5$. P, PT, CM, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant mean, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1900 to 02/2016.

	Good Volatility		Bad Volatility		Good Kurtosis		Bad Kurtosis	
	Exp.	Rec.	Exp.	Rec.	Exp.	Rec.	Exp.	Rec.
CM-CV	2.28%	3.61%	2.46%	4.79%	2.11	2.04	3.66	13.24
P-CV	21.68%	59.06%	13.32%	25.02%	101.32	46.15	7.05	0.26
PT-CV	23.43%	68.08%	13.87%	25.13%	114.96	68.96	7.89	0.20
P-SV	20.18%	38.11%	12.63%	26.04%	94.58	49.65	4.68	8.81
PT-SV	24.60%	43.60%	15.78%	25.85%	54.46	73.08	2.46	5.99

Table 12: Out-of-Sample Ratio of Certainty Equivalent Returns.
CER, Unc, Exp., and Rec. stand for certainty equivalent return, unconditional, expansion, and recession, respectively. The investment horizon is $T = 1$ and risk aversion is $\gamma = 5$. P, PT, CV, and SV stand for permanent component with stochastic drift, permanent component with stochastic drift and transitory component, constant volatility, and stochastic volatility, respectively. Statistics are computed using monthly S&P 500 data from 02/1900 to 02/2016.

	PT-CV/P-CV			PT-SV/P-SV		
	Unc.	Exp.	Rec	Unc.	Exp.	Rec
Ratio of Mean CER	1.0214	1.0142	1.0414	1.5898	1.6679	1.3849
Ratio of Median CER	0.9839	0.9787	1.0091	1.2072	1.2064	1.2119

(Engle, 1982; Bollerslev, 1986), our results show that ignoring the transitory component of returns, and therefore mismodeling the observed term structure of equity returns, is very costly to investors. Furthermore, Table 12 shows that the benefits of modeling the transitory component tend to be concentrated in recessions. The reason is the term structure of equity returns are more downward-sloping in recessions than in expansions. Therefore, the benefits of a model able to replicate this negative slope relative to one that cannot are particularly large in recessions.

6 Conclusion

This paper provides evidence that properly modeling the shape of the term structure of equity returns significantly increases investors' portfolio performance. We assume that equity returns are driven by a permanent component with stochastic drift and volatility, and a transitory component. In contrast to existing models considering mean-reverting expected returns and stochastic volatility, ours is flexible enough to generate either flat, or upward-sloping, or downward-sloping term structures of equity return volatility and risk premia. Fitting our model to the data by the method of maximum likelihood provides parameters that replicate the observed downward-sloping term structures of equity returns. Both in-sample and out-of-sample, trading strategies that account for the observed shape of the term structure of equity returns significantly outperform those which do not. Indeed, properly modeling the behavior of short-term equity returns implies surges in portfolio returns, which significantly increase the strategy's certainty equivalent return. Finally, we show that out-performance is concentrated in recessions, periods in which the term structures of equity returns are the most downward-sloping.

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A Appendix

A.1 Proof of Proposition 2

The optimal value function is of the form $J(t, W, x, z)$, and the HJB equation is as follows.

$$J_t - \kappa_z z J_z + \frac{1}{2}\eta_z^2 J_{zz} - \kappa_x x J_x + \frac{1}{2}\eta_x^2 J_{xx} + \eta_z \eta_x J_{zx} + \sup_{\pi} \Big\{ r_f W J_W + \pi W \left(m + x - \kappa_z z - r_f\right) J_W + \frac{1}{2}\pi^2 W^2 \bar{\upsilon} J_{WW} + \pi W \sqrt{\bar{\upsilon}} \eta_x J_{Wx} + \pi W \sqrt{\bar{\upsilon}} \eta_z J_{Wz} \Big\} = 0, \quad (24)$$

where

$$\eta_x \equiv \frac{\overline{\gamma}_x - \kappa_z \overline{\gamma}_{zx}}{\sqrt{\bar{\upsilon}}}, \quad \eta_z \equiv \frac{\overline{\gamma}_{zx} - \kappa_z \overline{\gamma}_z + \sigma_z^2}{\sqrt{\bar{\upsilon}}}$$

and the steady-state posterior variance-covariance matrix

$$\overline{\Gamma} \equiv \begin{pmatrix} \overline{\gamma}_z & \overline{\gamma}_{zx} \\ \overline{\gamma}_{zx} & \overline{\gamma}_x \end{pmatrix}$$

is obtained by solving the following system of equations $\frac{d\gamma_{z,t}}{dt} = 0$, $\frac{d\gamma_{x,t}}{dt} = 0$, and $\frac{d\gamma_{zx,t}}{dt} = 0$.

Solving the first order condition yields the following optimal portfolio weight

$$\pi^\star = -\frac{J_W}{W J_{WW}} \frac{m + x - \kappa_z z - r_f}{\bar{\upsilon}} - \frac{J_{Wx}}{W J_{WW}} \frac{\eta_x}{\sqrt{\bar{\upsilon}}} - \frac{J_{Wz}}{W J_{WW}} \frac{\eta_z}{\sqrt{\bar{\upsilon}}}. \quad (25)$$

Inspired by results in Kim and Omberg (1996) and Zariphopoulou (2001), we make the conjecture

$$J(t, W, x, z) = e^{-\delta t} \frac{W^{1-\gamma}}{1-\gamma} F(t, x, z)^{\gamma}. \quad (26)$$

Plugging this conjecture back into the HJB equation (24) implies that the function $F(t, x, z)$ solves the linear second-order PDE

$$F_t + \left[-\kappa_x x + \frac{(1-\gamma)(m + x - \kappa_z z - r_f)\eta_x}{\gamma\sqrt{\bar{\upsilon}}}\right] F_x + \left[-\kappa_z z + \frac{(1-\gamma)(m + x - \kappa_z z - r_f)\eta_z}{\gamma\sqrt{\bar{\upsilon}}}\right] F_z$$

$$+ \frac{1}{2}\eta_x^2 F_{xx} + \frac{1}{2}\eta_z^2 F_{zz} + \eta_x \eta_z F_{xz} + \frac{1-\gamma}{\gamma}\left[r_f - \frac{\delta}{1-\gamma} + \frac{\left(m + x - \kappa_z z - r_f\right)^2}{2\gamma\bar{\upsilon}}\right] F = 0. \quad (27)$$

The solution to PDE (27) is given by

$$F(\tau, \theta) = \exp\left(\frac{1}{2}\theta^\top A(\tau)\theta + \theta^\top B(\tau) + C(\tau)\right), \quad (28)$$

where $\theta = (x, z)^\top$, $A(\tau)$ is a symmetric 2×2 matrix-valued function, $B(\tau)$ is a 2×1 matrix-valued function and $C(\tau) \in \mathbb{R}$ with A, B, and C solving the system of matrix Riccati differential equations

$$\begin{aligned}
\dot{A}(\tau) &= \frac{1}{2} A(\tau) q_1^\top q_1 A(\tau) + (q_2 + q_3 q_1) A(\tau) + A(\tau)(q_2 + (q_3 q_1)^\top) + q_4 \\
\dot{B}(\tau) &= \frac{1}{2} A(\tau) q_1^\top q_1 B(\tau) + (q_2 + q_3 q_1) B(\tau) + A(\tau) q_1^\top q_5 + q_6 \\
\dot{C}(\tau) &= \frac{1}{4} B^\top(\tau) q_1^\top q_1 B(\tau) + \frac{1}{2} q_7^\top A(\tau) q_7 + B(\tau)^\top q_1^\top q_5 + q_8
\end{aligned}$$

with coefficients

$$\begin{aligned}
q_1 &= \begin{pmatrix} \eta_x & \eta_z \\ \eta_x & \eta_z \end{pmatrix}, \quad q_2 = \begin{pmatrix} -\kappa_x & 0 \\ 0 & -\kappa_z \end{pmatrix}, \\
q_3 &= \frac{1-\gamma}{\gamma} \frac{1}{\sqrt{\bar{v}}} \begin{pmatrix} 1 & 0 \\ 0 & -\kappa_z \end{pmatrix}, \quad q_4 = \frac{1-\gamma}{\gamma} \frac{1}{\gamma \bar{v}} \begin{pmatrix} 1 & -\kappa_z \\ -\kappa_z & \kappa_z^2 \end{pmatrix}, \\
q_5 &= \frac{1-\gamma}{\gamma} \frac{(m - r_f)}{\sqrt{\bar{v}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad q_6 = \frac{1-\gamma}{\gamma} \frac{(m - r_f)}{\gamma \bar{v}} \begin{pmatrix} 1 \\ -\kappa_z \end{pmatrix}, \\
q_7 &= \begin{pmatrix} \eta_x \\ \eta_z \end{pmatrix}, \quad q_8 = \frac{1-\gamma}{\gamma} \left[r_f - \frac{\delta}{1-\gamma} + \frac{(m - r_f)^2}{2\gamma \bar{v}} \right]
\end{aligned}$$

The boundary conditions are $A_{ij}(0) = B_i(0) = C(0) = 0$, where A_{ij} and B_i denote the components of A and B. Substituting Equation (28) in Equation (26) and plugging the result in Equation (25) yields the optimal portfolio weight.

□

A.2 Proof of Proposition 3

For the problem with stochastic volatility v_t, the optimal value function is of the form $J(t, W, x, z, v, \gamma_x, \gamma_z, \gamma_{zx})$, and the HJB equation is as follows.

$$
\begin{aligned}
& J_t - \kappa_x x J_x - \kappa_z z J_z + \kappa_v(\bar{v} - v) J_v \\
& + \left(\sigma_x^2 - 2\kappa_x\gamma_x - \frac{1}{v}\left(\gamma_x - \gamma_{zx}\kappa_z\right)^2\right) J_{\gamma_x} + \left(\sigma_z^2 - 2\kappa_z\gamma_z - \frac{1}{v}\left(\gamma_{zx} - \gamma_z\kappa_z + \sigma_z^2\right)^2\right) J_{\gamma_z} \\
& + \left(-\left(\kappa_x + \kappa_z\right)\gamma_{zx} - \frac{1}{v}\left(\gamma_x - \gamma_{zx}\kappa_z\right)\left(\gamma_{zx} - \gamma_z\kappa_z + \sigma_z^2\right)\right) J_{\gamma_{zx}} \\
& + \frac{1}{2v}(\gamma_{zx} - \kappa_z\gamma_z + \sigma_z^2)^2 J_{zz} + \frac{1}{2v}(\gamma_x - \kappa_z\gamma_{zx})^2 J_{xx} + \frac{1}{2}\sigma_v^2 v J_{vv} \\
& + \frac{1}{v}\left(\gamma_{zx} - \kappa_z\gamma_z + \sigma_z^2\right)\left(\gamma_x - \kappa_z\gamma_{zx}\right) J_{zx} + (\gamma_x - \kappa_z\gamma_{zx})\sigma_v J_{xv} + (\gamma_{zx} - \kappa_z\gamma_z + \sigma_z^2)\sigma_v J_{zv} \\
& + r_f W J_W + \sup_\pi \Big\{ \pi W\left(m + x - \kappa_z z - r_f\right) J_W + \frac{1}{2}\pi^2 W^2 v J_{WW} + \pi W(\gamma_x - \kappa_z\gamma_{zx}) J_{Wx} \\
& + \pi W(\gamma_{zx} - \kappa_z\gamma_z + \sigma_z^2) J_{Wz} + \pi W v \sigma_v J_{Wv} \Big\} = 0, \qquad (29)
\end{aligned}
$$

Solving the first order condition yields the following optimal portfolio weight

$$
\begin{aligned}
\pi^\star = & - \frac{J_W}{W J_{WW}} \frac{(m + x - \kappa_z z - r_f)}{v} - \frac{J_{Wx}}{W J_{WW}} \frac{\gamma_x - \kappa_z\gamma_{zx}}{v} \\
& - \frac{J_{Wz}}{W J_{WW}} \frac{\gamma_{zx} - \kappa_z\gamma_z + \sigma_z^2}{v} - \frac{J_{Wv}}{W J_{WW}} \sigma_v
\end{aligned} \qquad (30)
$$

After plugging Equation (30) into the HJB equation (29), and guessing that

$$
J(t, W, x, z, v_t, \gamma_x, \gamma_z, \gamma_{zx}) = e^{-\delta t} \frac{W^{1-\gamma}}{1-\gamma} F(T - t, x, z, v, \gamma_x, \gamma_z, \gamma_{zx})^\gamma
$$

we obtain a nonlinear PDE for the F function. We look for an approximate analytic solution to this PDE of the form

$$
F(\tau, \theta) = \exp\left(\frac{1}{2}\theta^\top A(\tau)\theta + \theta^\top B(\tau) + C(\tau)\right), \qquad (31)
$$

where $\theta = (x, z, v, \gamma_x, \gamma_z, \gamma_{zx})^\top$, $A(\tau)$ is a symmetric 6×6 matrix-valued function, $B(\tau)$ is a 6×1 matrix-valued function and $C(\tau) \in \mathbb{R}$. Plugging in (31) in the PDE and performing a second order Taylor expansion around the long-run means of the state variables allows us to use the variable separation method to obtain a coupled system of ODEs for coefficient functions A, B, and C.

□

A.3 Proof of Proposition 4

In the model with permanent component only, the true (full observation) moment generating function of the logarithm of the stock price is given by

$$MGF(t,\tau,u) = E\left[S_{t+\tau}^u|\mathcal{F}_t\right] = e^{u\log(S_t)+h_0(\tau;u)+h_1(\tau;u)x_t}, \tag{32}$$

where h_0 and h_1 are deterministic functions of time solving a system of ODEs

$$\begin{aligned} h_1'(\tau) &= u - \kappa_x h_1(\tau), \\ h_0'(\tau) &= \frac{1}{2}\overline{v}u^2 + u(m - \frac{\overline{v}}{2}) + \frac{1}{2}\sigma_x^2 h_1^2(\tau). \end{aligned}$$

Similarly, the filtered (partial observation) moment generating function of the logarithm of the stock price is given by

$$\widehat{MGF}(t,\tau,u) = E\left[S_{t+\tau}^u|\mathcal{F}_t^S\right] = e^{u\log(S_t)+\widehat{h}_0(\tau;u)+\widehat{h}_1(\tau;u)\widehat{x}_t}, \tag{33}$$

where $\widehat{h}_0$ and $\widehat{h}_1$ solve

$$\begin{aligned} \widehat{h}_1'(\tau) &= u - \kappa_x \widehat{h}_1(\tau) \\ \widehat{h}_0'(\tau) &= \frac{1}{2}\overline{v}u^2 + u\left(m - \frac{\overline{v}}{2} + \overline{\gamma}_x\widehat{h}_1(\tau)\right) + \frac{1}{2}\frac{\overline{\gamma}_x^2}{\overline{v}}\widehat{h}_1^2(\tau) \end{aligned}$$

Consequently, (22)–(23) follow directly from (32)–(33) . Taking derivative of (22) with respect to the horizon τ gives

$$\partial_\tau \sigma_S(t,\tau)^2 = \frac{e^{-2\kappa_x\tau}\left(1 + 3e^{2\kappa_x\tau} + 2\kappa_x\tau - 4e^{\kappa_x\tau}(1+\kappa_x\tau)\right)\sigma_x^2}{2\kappa_x^3\tau^2}. \tag{34}$$

As $\kappa_x \geq 0$, to determine the sign of the partial derivative we need to determine the sign of the numerator in (34). Notice that it is 0 at $\tau = 0$ and nondecreasing in τ as

$$\begin{aligned} &\frac{\partial}{\partial\tau}\left(e^{-2\kappa_x\tau}\left(1 + 3e^{2\kappa_x\tau} + 2\kappa_x\tau - 4e^{\kappa_x\tau}(1+\kappa_x\tau)\right)\sigma_x^2\right) \\ &\quad = 4e^{-2\kappa_x\tau}(-1+e^{\kappa_x\tau})\kappa_x^2\sigma_x^2\tau \geq 0. \end{aligned}$$

Similarly, for (23) we have

$$\partial_\tau \sigma_S(t,\tau)^2 = 2\frac{e^{-\kappa_x\tau}(-1+e^{\kappa_x\tau}-\kappa_x\tau)\overline{\gamma}_x}{\kappa_x^2\tau^2} + \frac{e^{-2\kappa_x\tau}(1 + 3e^{2\kappa_x\tau} + 2\kappa_x\tau - 4e^{\kappa_x\tau}(1+\kappa_x\tau)\overline{\gamma}_x^2}{2\overline{v}\kappa_x^3\tau^2}.$$

The numerator is 0 at $\tau = 0$ and nondecreasing in τ as

$$4e^{-2\kappa_x\tau}\left((-1+e^{\kappa_x\tau})\overline{\gamma}_x + e^{\kappa_x\tau}\overline{v}\kappa_x\right)\kappa_x^2\overline{\gamma}_x\tau \geq 0.$$

□